DOCUMENT RESUME

ED 222 338 SE 039 383

TITLE Critical Reviews in Mathematics Education. Materialien und Studien, Band 9.
INSTITUTION Bielefeld Univ. (West Germany). Inst. for Didactics in Mathematics.
PUB DATE 79
NOTE 139p.

EDRS PRICE MF01/PC06 Plus Postage.
DESCRIPTORS *Educational Research; Elementary Secondary Education; Fractions; Geometric Concepts; Geometry; Instruction; *Mathematics Curriculum; *Mathematics Education; *Mathematics Instruction; *Mathematics Teachers; *Teaching Methods
IDENTIFIERS *Mathematics Education Research

ABSTRACT

Four papers are presented which view research in mathematics education from different perspectives. The titles are: (1) Review of Recent Research Related to the Concepts of Fractions and of Ratio; (2) Some Trends in Research and the Acquisition and Use of Space and Geometry Concepts; (3) A Portrayal of Traditional Teachers of Mathematics in American Schools; and (4) Hidden Dimensions in the So-Called Reality of a Mathematics Classroom. The first three are related to invited presentations at the second meeting of the International Group for Psychology and Mathematics Education (IGPME) in Osnabruck in September 1978. The fourth was developed as an invited address at the annual meeting of the Special Interest Group for Research in Mathematics Education (SIG/RME), which was run as a pre-session to the National Council of Teachers of Mathematics (NCTM) conference in Boston in April 1979. (MP)

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Critical Reviews in Mathematics Education

Materialien und Studien, Band 9

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Fotovorlage: Maria Otte
Hans Kock Buch- und Offsetdruck GmbH, Bielefeld

PREFACE

Research in mathematics education under different perspectives is the theme of the four papers collected in this issue. The first three are related to invited presentations at the second meeting of the International Group for Psychology and Mathematics Education (IGPME, an official subgroup of ICME) in Osnabrück, September 4 - 9, 1978. The IDM had animated and sponsored Marilyn N. Suydam's report on research related to the learning and teaching of fractions and ratios, Richard Lesh's report on spatial and geometrical concepts, and Jack A. Easley's report about his (jointly with Robert Stake directed) case studies in science education, particularly about the aspects related to mathematical education in these studies. The fourth paper was developed for an invited address at the annual meeting of the Special Interest Group for Research in Mathematics Education (SIG/RME in AERA), which was run as pre-session to the NCTM conference in Boston on April 18, 1979.

Heinrich Bauersfeld, June 1979

VORWORT

Forschung zum Mathematikunterricht unter verschiedenen Perspektiven kennzeichnet die in diesem Band vereinigten Vorträge. Die ersten drei gehen auf Einladungen zur 2. Tagung der Internationalen Studiengruppe für Psychologie und Mathematikunterricht (IGPME, eine offizielle Untergruppe der Internationalen Mathematischen Unterrichtskonferenz) in Osnabrück vom 4. - 9. September 1978 zurück. Das IDM hat die Forschungsberichte von Marilyn N. Suydam über das Lehren und Lernen von Bruch- und Verhältnisbegriff, von Jack A. Easley über seine (zusammen mit Robert Stake durchgeführten) Fallstudien zum Naturwissenschaftlichen Unterricht, insbesondere über die darin einbezogenen Untersuchungen zum Mathematikunterricht, und von Richard Lesh über Raumanschauung und geometrische Grundbegriffe angeregt und gefördert. Der vierte Beitrag wurde für eine Einladung zur Jahreskonferenz der Special Interest Group for Research in Mathematics Education (SIG/RME, eine Untergruppe der Amerikanischen Gesellschaft für Erziehungswissenschaftliche Forschung, AERA) geschrieben, die als Vorkonferenz zur NCTM-Jahrestagung (National Council of Teachers of Mathematics) am 18. April 1979 in Boston stattfand. (Die Literaturhinweise sind dementsprechend ausgewählt.)

Heinrich Bauersfeld, Juni 1979

CONTENTS

MARILYN N. SUYDAM

Review to the Concepts of Fraction and of Ratio 3 - 45

RICHARD LESH

Some Trends in Research and the Acquisition and Use of Space and Geometry Concepts ..... 46 - 83

JACK EASLEY

A Portrayal of Traditional Teachers of Mathematics in American Schools ..... 84 - 108

HEINRICH BAUERSFELD

Hidden Dimensions in the So-Called Reality of a Mathematics Classroom ..... 109 -

REVIEW OF RECENT RESEARCH RELATED TO THE CONCEPTS OF FRACTIONS AND OF RATIO

Marilyn N. Suydam

Discussions about and reviews of research on fractions or rational numbers have been relatively extensive during the past several years. Kieren (1976a) elaborated the mathematical, cognitive, and instructional foundations of rational numbers in one of a set of papers from a research workshop sponsored by the Georgia Center for the Study of Learning and Teaching Mathematics (GCSLTM). For the same meeting, Payne (1976) described the studies which he, and graduate students working with him at The University of Michigan, had conducted over a period of years. For the Third International Congress on Mathematical Education, Kieren (1976b) briefly summarized both previous studies and ongoing research being conducted by the Rational Number Working Group of the GCSLTM. Other papers provide insights on selected aspects of research or assessment efforts.

Reviews by Suydam and Dessart (1976) and Suydam and Weaver (1975) have summarized the results of a wide range of studies on fractions, with a focus on presenting findings which might have relevance for teachers. Instructional procedures, many concerned with computational algorithms for fractions, have characterized such studies.

In another approach, procedures which have been developed through research at The University of Michigan are carefully detailed for teachers. Thus Coxford and Ellerbruch (1975) describe a sequence for developing the fraction concepts; this is refined, with even more detail, by Ellerbruch and Payne (1978).

Far less attention has been devoted to ratio concepts aside from their involvement in interpretations of fractions. While some attention has been given to the curricular aspects of teaching ratio and proportion, most research has explored the topic in relation to statements about proportionality made by Piaget.

This paper will not attemtpt to describe myriad studies in detail. Instead, highlights from previous documents will be cited to indicate the status of our knowledge about fraction and ratio concepts and operations, to draw attention to comments from researchers on their findings, and to note points to consider as research proceeds.

Assessment Results on Fractions

Data from the first mathematics assessment of the National Assessment of Educational Progress indicate that performance with fractions is at a much lower level than performance with whole numbers (Carpenter et al., 1978). In most American schools, some intiutive ideas about fractions are introduced in the primary grades; however, it is "not too likely that they will generally think of fractions as quantities" (p. 34). In-depth study of fraction concepts, operations with fractions, and emphasis on fractions as quantities occurs in the last half of grade 4, and continues into grades 5 and 6, with review and reinforcement continuing in grades 7 and 8. Consequently,

> 9-year-olds (grades 3 and 4) assessed in January and February would not be expected to demonstrate much formal knowledge of fraction concepts and algorithms, although 13-year-olds should be thoroughly operational with fractions. (Carpenter et al., 1978, p. 34)

In their overview of the findings on the data on fraction items, Carpenter et al. noted :

- Only 2o to 37 percent of the 9-year-olds could name a fractional part of a whole, and only 9 to 21 percent were correct on naming a fractional part of a set of discrete elements. There appears to be an absence of fraction concepts for 9-year-olds, rather than any erroneous knowledge for the majority of this age group.

- While 65 percent of the 13-year-olds were correct in naming the fractional part of a set of discrete elements, they and adults performed at about the same level (2o and 25 percent respectively) when naming the fractional part of a group, with 17-year-olds slightly better (36 percent). Thus, the overall results were low.

- None of the age groups appear to possess adequate knowledge of properties of fractions, especially ordering and comparing. Fifty-six percent of the 13-year-olds and 83 percent of the 17-year-olds could select a fraction between two given fractions. No age group could order adequately six fractions from smallest to largest; the respective percentages correct were 19, 39, and 36, respectively.

- A majority of 13- and 17-year-olds could successfully multiply 1/2 x 1/4 (62 and 74 percent respectively), although fewer (42 and 66 percent respectively) were able to add 1/2 and 1/3 correctly. On an addition problem, 30 percent of the 13-year-olds and 16 percent of the 17-year-olds added numerators and added denominators. This indicates a lack of understanding of fractions as quantities and a severe failure to develop computational skill or even knowledge of computation algorithms.

- On an exercise involving a minor translation to a number sequence, selection of the multiplication operation, and multiplication of a fraction by a whole number, 78 percent of the 17-year-olds, 57 percent of the 13-year-olds, and 17 percent of the 9-year-olds could do the task.

- Another exercise called for the manipulation of an algebraic fraction x/y. While it was administered to 9-year-olds, it is difficult to see any reasonable expectation for them to cope with the symbolism and understand the problem; nevertheless, 15 percent of them hit upon the correct answer. Eighteen percent of the 13-year-olds and 41 percent of the 17-year-olds were correct.

In a selection of implications for instruction, Carpenter et al. (1978) noted:

> Emphasis should be placed on providing a sound initial development of fraction concepts using concrete objects. Results indicate that many students have little computational skill with fractions and probably little conceptual understanding. An increase in the amount of time spent on operations with fractions is not necessarily an appropriate remedy. The development of algorithms should be paced so as to connect firmly with the main ideas in the initial development. (p. 54)

Bright (1978) synthesized results from NAEP, NLSMA, and several other assessments. He reported that

> Performance seems to stabilize at about grade 8 (age thirteen), and there is noticeable improvement in performance in grade 6 to grade 8. As might be expected when the lowest common denominator does not appear in the problem students have considerably more trouble than when it does appear.Their skills with multiplying fractions seem to lie somewhere between these two levels. It is impossible to determine from the data, however, whether their difficulties result from a lack of understanding of the processes, an inability or inattention to "reducing" the answers to lowest terms, or a lack of practice. (p. 158)

> It should be noted in passing that the multiplication algorithm, because it can be performed without writing the fractions with a common denominator, is mechanically an easy one. Conceptually, however, multiplying fractions is more complex than adding them ... this hierarchy of difficulty is supported empirically. Apparently the mechanical difficulties of adding fractions without a common denominator outweigh the conceptual difficulties of multiplication. (p. 159)
>
> ... stabilization of performance of whole-number computation occurs earlier and at a higher level than for fractional-number computation. Perhaps this is a practice effect that reflects the introduction of whole-number computation before fractional-number computation. (p. 160)

Suydam and Osborne (1977) also summarized data from various assessments, concluding that "the topics with which difficulty (or weakness) were reported can be ranked in this order of frequency: first, fractions ..." (p. 212).

From interviews with 176 seventh graders, Lankford (1972) reported errors made on fraction examples. Of the possible answers for computation with fractions, 35 percent were right, 33 percent were wrong, and 32 percent were omitted. It was observed that performance with fractions was much below that with whole numbers. He noted the most prevalent incorrect practices.

Addition

1. Adding numerators and placing the sum over one of the denominator or over a common denominator.
2. Adding numerators for the numerator of the sum and the same for the denominators.
3. Writing equivalent fractions with common denominators.
4. Adding the numerator and denominator of one fraction for the numerator of the sum, and the same with the second fraction for the denominator of the sum,

Subtraction

1. Subtracting numerators for the numerator of the difference and the same with denominators.

2. Using borrowing, or the borrowed number, incorrectly.

3. Writing equivalent fractions by dividing a denominator into the common denominator or adding this quotient to the numerator of the original fraction for the numerator of the equivalent fractions.

4. Writing equivalent fractions by choosing a common denominator and using it for the denominator of the new fraction but retaining the numerator of the old fraction.

Multiplication

1. First writing equivalent fractions, unnecessarily, then incorrectly multiplying numerators and placing the product over the common denominator.

2. Introducing errors with conversions to simpler form.

3. Writing the reciprocal of the second factor before multiplying.

4. In a mixed number times a fraction, multiplying the fractions and affixing the whole number.

5. In a mixed number times a whole number, multiplying the whole numbers and affixing the fraction.

Division

1. Dividing numerators and placing the product over the common denominator.

2. Making errors in dividing numerators.

3. In dividing a mixed number by a whole number, dividing the whole numbers and affixing the fraction.

4. In dividing a mixed number by a fraction, dividing the fractions and affixing the whole number.

5. Multiplying numerators of like fractions instead of dividing.

6. Multiplying numerators and denominators without writing the reciprocal of the divisor.

Of the responses on eight comparison exercises with fractions, 35 percent were right answers, 17 percent were wrong, and 48 percent were omitted. The errors included:

1. The fraction and the 1 were incorrectly compared.
2. Whichever divisor was thought to be larger made the corresponding sum, difference, product, or quotient larger.
3. Attempting to perform the operations before making the comparison.
4. Comparing the sums, etc., incorrectly.
5. Thinking of all whole numbers as greater than fractions.

Lankford includes all of the responses made by the students, providing a rich source of data which can serve for researchers (and teachers) as a guide. The sometimes tortuous explanations which the students use to describe their manipulations of numbers indicate clearly weaknesses in their understanding of the rational number construct.

As Ginther et al. (1976) noted,

> Our present instructional program does not provide our students, even in the best of schools, with the fractional skills, in computation and in problem solving, that they should have. (p. 10)

They conducted a study to investigate whether deficiencies in fraction skills are due to current instructional programs. A 78-item battery of fraction tests were developed; tests were classified according to cognitive level as computation, comprehension, or applications. More than 1500 average eight-grade students in programs considered to be "modern" took 10-item parts of the battery. Results indicated that

> students do not have the fractional skills needed to compute or to solve problems. Responses to number line and diagrammatic items indicate they understand fractional concepts; however, performance on other items indicates a poor understanding of the structure of the rational number system.

Thus, the data appear to indicate that even instruction purportedly aimed at developing understanding of the rational number construct does not necessarily result in better performance. Such assessment results have spurred research. They have led, on the other hand, to recommendations that fractions be de-emphasized or deleted from the curriculum, with arguments frequently pointing to increased use of the metric system or calculators and thus decreased need for fractions. Others, noting the long history of poor results with achievement on fractions, have concluded that instruction on rational numbers should be postponed until the learner is more mature. For the past two decades, the appropriate time for rational number instruction has seemed to be when the child reaches the Piagetian stage of formal operations (ages 11-14). Earlier, the investigations of the Committee of Seven (Washburne, 1931; Gillet, 1931) led to delineation of the mental ages at which success could be expected (according to the Committee's standard of 80 percent of the total number of examples in the retention test solved correctly by 75 percent of the group); they also listed "optimum mental ages", "at which the curve definitely flattened, indicating that there was little to be gained by postponement":

	Minimum M.A.	Optimum M.A.
Meaning of fractions	9-0	10-9
Addition and subtraction of similar fractions and mixed numbers, with no carrying	9-10	11-1
Multiplication of fractions	12-3	14-2
Division of fractions	12-3	

On a very different basis, Freudenthal (1973) argued that teaching of the addition algorithm for rational numbers should be delayed until it can be developed as a consequence of the algebraic ideas from which it arises. The student should be given the opportunities to deduce rational number ideas from more general ones.

As researchers have considered the process of learning fractions and rational number ideas, however, the emphasis has not been on postponement, but on sequencing and on clarification of the construct itself.

The Psychological Foundation of Rational Numbers

It has long been recognized that

> The fraction concept is complex and cannot be grasped all at once. It must be aquired through a fairly long process of sequential development. (Hartung, 1958, p. 377)

In commenting on the child's difficulty with rational numbers, Kieren (1976b) noted:

> A child may through natural experience attain certain "fractional" and part-whole notions. Yet, attaining a conceptualization of rational numbers requires a sophisticated application of the proportionality schema, a general notion of inverses, the ability to do arithmetic with

> two distinct abstractly defined operations, and some concept of one dimensional vections (addition) and function composition (multiplication). (pp. 1-2)

Compounding the difficulty is the fact that rational numbers have many interpretations:

1. Rational numbers are fractions which can be compared, added, subtracted, etc.
2. Rational numbers are decimal fractions and form a natural extension of the whole numbers.
3. Rational numbers are equivalence classes of rational numbers.
4. Rational numbers are numbers of the form p/q, where p and q are integers, that is "ratio" numbers.
5. Rational numbers are multiplicative operators, states related by such operators, or dilatations.
6. Rational numbers are elements of a quotient field.
7. Rational numbers are measures or points on a number line. (Kieren, 1976b, p. 2)

Kieren noted that the aquisition of whole-number concepts is "aided by everyday experiences with both precede and accompany formal instruction" (p. 2). Such experience with rational numbers, however, are not particularly common. He also stated that

> While there seems an obvious relationship between the aquisition of the proportionality schema and rational number aquisition, the nature of this has only been explored in a limited fashion. Relationship to other fundamental intellectual structures such as the INRC group structures are hypothetical at this stage. Suffice it to say then that a "natural" development sequence for the concept of rational numbers is not known at this stage. (p. 3)

Thus, in working with rational numbers, "children are dealing with mathematical structures which do not have an

obvious basis in natural thought" (Kieren, 1976a, p.103).

He also discussed the little that formal instruction has done to "enrich the experience base of children". Instruction has focused on the first two of his interpretations, ignoring the remainder. He believes that

> Because each interpretation of rational relates to particular cognitive structures, ignoring a conglomerate picture or failing to identify particular necessary structures in developing instruction can lead to a lack of understanding on the part of the child. (Kieren, 1976a, p. 127)

He makes the point that

> the many interpretations of rational numbers have themselves many related instructional strategies. These in turn employ numerous physical and symbolic models. Without a conglomerate view, it is easy to design instructional settings which contain contradictory elements or models, or which do not easily lead to the development of some rational number concepts. For example, if one interprets rationals as measures and uses a number line model, multiplication of rationals is not naturally generated. The number line model may conflict cognitively with an area model or an exchange model for generating multiplicative ideas. (Kieren, 1976a, p. 127)

Quite obvious should be the fact that he proposes that curriculum developers must consider both short- and long-term objectives for instruction on rational numbers, relate these objectives with the interpretations of rational numbers, and then select appropriate interpretations to develop certain objectives.

> Given these, they can then ascertain the necessary cognitive structures for meeting the objectives and develop sequences of instructional activities which contribute to the growth of these structures. (Kieren, 1976a, p. 128)

Researchers must use a similar process to study selected interpretations, identify the most important cognitive structures, develop settings to ascertain the extent to which a child has such structures, and study the growth of such structures developmentally.

Kieren is, obviously, attempting to devise a schema for improving research and development efforts. Thus he noted:

> The seven interpretations noted above each relate to or are suggestive of schema or cognitive structures which at least theoretically appear to underlie rational number concept development. Further, these mathematical structures and related cognitive structures then induce a set of related instructional structures. While our theory is yet lacking in formal propositions, an example of the relationship between mathematical, cognitive and instructional structures is given below:
>
> Rational numbers notions conceptualized as operators have as some of their underlying cognitive structures the proportionality schema, a generalized notion of reversibility, conservation of quantity under equi-sized decomposition, composition of transformation and the conservation of results under such a composition and some or all of the INCR group structures. (From a developmental point of view it is clear that such rational number notions could have several levels or stages.)
>
> The mathematical and cognitive structures above could be developed using the following instructional structures: measurement division problems, proportional exchange settings, or similarity mapping settings. (pp.3-4)

In another analysis of the fraction concept, Wagner (1976) underlined the definitional problem in another way, indicating

> the need for looking at the fraction concept as a complex "megaconcept" made up of a variety of components or subconcepts ... not sequential in nature. Instead, the development of the fraction concept can be visualized as strands

> of subconcepts that become increasingly more intertwined, each enhancing the understanding of the other, until they become so interwoven that they are one. In addition, strands from other suject areas (mathematical and otherwise) become intermeshed, contributing understanding to and deriving understanding from the developing fraction concept ... This development was viewed as a gradual process of internal construction. The culmination, which involves an understanding of the isomorphic character of all the models of fractions, is the concept of the system of rational numbers.(pp.7920-7921)

Identification of the attributes of fractional parts has been of some concern to researchers. Piaget, Inhelder, and Szeminska (1960) indicated that a child's concept of fractional parts is based on seven attributes:

(1) a whole is composed of separable elements

(2) separation can occur into a determinate number of parts

(3) subdivision exhausts the whole

(4) a fixed relationship exists between dividing cuts and subdivisions

(5) subdivisions are equal

(6) these parts are wholes in their own right

(7) the whole is conserved

They found that children well into elementary school had not attained these attributes. Payne (1976) verified this and added other components which he sees as essential to early rational number learning:

(8) symbolic control over fractions

(9) continuous and discrete part-whole relationships

(10) fractions larger than one

(11) equivalent subdivisions

Novillis (1976), working with number lines, added:

(12) ability to identify the unit

Kieren (1976a) suggested:

(13) arbitrary subdivision of the unit

(14) proportionality

(15) composition of transformation

(16) control over two-dimensional symbols

(17) control of primitive forms of vector addition and function composition

(18) functional capability with equivalence classes and quotient fields

(19) connections with natural and real numbers

It has also been apparent to these researchers that children must have experiences in a variety of contexts. Some of the specific results of their research are included in the next section.

Research on Instruction on Fractions

As was noted in the introduction, Suydam and Dessart 1976) and Suydam and Weaver (1975) have provided reviews of the research on fractions with a particular emphasis on studies which may have direct implications for classroom practice. These reviews provide citations to studies which have tended to be concerned with computational skills with fractions. Instead of duplicating these syntheses, attention here will focus largely on studies in which the researchers were concerned with the processes children use as they learn fractional concepts and develop skills with the operations on fractions.

Payne (1976) reviewed the studies completed in two periods of work at The University of Michigan (with additional

comments on other related research).

Period 1:	Emphasis on comparing and analyzing the strengths and weaknesses of algorithms for a given fraction operation		
Bidwell	1968	division of fractions	comparing three approaches (common denominator, complex fraction, and inverse operation)
Green	1969	multiplication of fractions	comparing two approaches (area and finding fractional parts of) and two modes (diagrams and manipulative materials)
Bohan	1970	equivalent fractions	comparing three sequences using diagrams or paperfolding
Coburn	1973	equivalent ratios	comparing two approaches (ratio and regions)
Period 2:	More intensive examination of what children learn while being taught a carefully developed sequence (study of the cognitive structures resulting from different combinations of rules and concrete models)		
Muangnapoe	1975	initial fraction concept and symbols	
Williams	1975	initial fraction concepts	
Galloway	1975	initial fraction sequence in grades 1-5	
Choate	1975	cognitive structures of children taught a rule for comparing fractions or a conceptual approach to comparison	
Ellerbruch	1975	effects of changing order of rules and concrete models for equivalent fractions, addition, and subtraction	

Among the comments made by Payne about the various studies, some have particular significance in indicating specific points learned:

- ... it is important to do an analysis of the various components of the learning structure for an algorithm. Performance on some major subskills needed in the development of an algorithm showed spots where more effective instructional material was needed. (p. 149)

- The Common Denominator method lacks internal structure compared with the Complex Fraction or Inverse Operation methods. (p. 149)

- It appears in retrospect that the use of visual materials in developing algorithms has a more important effect on retention than does a purely logical mathematical development. (p. 155)

- It is striking that the cognitive structure for pupils in the Area approach were superior to those in the approach Finding Fractional Parts. Since there are more applications and practical uses of FFP, what may be needed is an investigation of how this approach can be taught so that the cognitive structure has a better fit. (p. 155)

- Evidently it is much more complicated to relate a child's thought to his use of concrete materials and/or diagrams of the materials than is usually assumed. (p. 156)

- Paper folding helped make a logical connection between the concrete models and the generalization for getting higher terms. (pp. 162-163)

- Taking Bidwell, Green, and Bohan together, it appears algorithms were learned best when they were logical, and retention seemed best when there was a heavy visual component to instruction. (p. 163)

- Bohan's study demonstrated that instruction on operations on fractions can begin with multiplication. (p. 163)

- In choosing an initial model to be given main emphasis, one must look ahead to subsequent work. (p. 167)

- These data and other evidence point to a more consistent and sound cognitive structure being built using Region as the initial model. (p. 167)

- It seems likely that improvement ... in symbolic form might come by spending more time initially with concrete materials, or else more time is needed to relate the symbolic form to the concrete materials. (p. 167)

- Logical developments of algorithms [and] concrete objects that seemed to fit well with steps in the algorithms appeared to help achievement, and this was more pronounced on retention tests. (p. 167)

These comments, all reflections on work during the first period of studies, affected the research questions and procedures used in the later studies. These are characterized by extensive work with children in small groups to explore processes and refine sequences of instruction. Payne noted additional perceptions and conclusions from these studies:

- Teaching word names before fraction symbols eliminated the "reversal problem". Children were having perceptual difficulties with diagrams. (p. 169)

- The set model was dropped from the sequence because it seemed to interfere with the measurement ideas associated with regions and because achievement was low on the set model. Difficulties persisted with fractions associated with number lines ... (p. 171)

- All investigators found that the set model for fractions was difficult to teach. After teaching the set model ... , the region model with emphasis on measurement seemed to disintegrate. The set model is very closely related to ratio; in fact, they may be the same thing. Since ratio is a topic ... considered important, a way is needed to teach the initial set model and yet maintain the measurement model. (p. 179)

- What seems clearest from Choate's work is that steps in a rule developed side by side with visual diagrams is the least effective way to teach an algorithm ... there may be difficulty in teaching two seemingly different things at the same time. (p. 176)

- There is evidence that relating a spatial representation and a verbal algorithm helps in learning equivalent fractions. (p. 180)

- ... suggests that a ratio-proportion treatment for the initial work with equivalent fractions be delayed. (p. 180)

- This writer is convinced that there should be some spatial models, propably because they give better structures and also better retrieval from semantic memory. The way the rules and spatial models can and should be related is still a major question. (p. 181)

- Overall, the Michigan studies show that it takes a very much longer period of time to teach any part of the fraction work than has generally been allowed in instructional materials or curriculum guides. (p. 182)

It is vital to note that conflicts apparently arise when instruction based on different interpretations (such as ratio and measurement) are used together. Sambo (1975) found that textbooks for nine-to-eleven-year olds typically present fraction concepts in a variety of modes simultaneously, probably leading to confusion on the part of the children.

Owen (1975) tested children aged 8 and 9 on such pre-measurement attributes as conservation of length and area, and then instructing them using the Initial Fraction Sequence developed at Michigan.

> He found that for the 9-year-olds there was a significant difference in achievement of IFS objectives between students with high pre-measurement scores ... and those with low scores. This finding is suggestive of an hypothesis that attainment of the sturctures of the full stage of concrete operations may be a necessary condition for learning initial rational number concepts. (Kieren, 1976b, p. 9)

Kieren also notes that several studies are attempting to establish further the connection between measurement attributes and number learning (e.g., Babcock, 1976; Sambo, 1976).

In outlining related and continuing work, Kieren (1976b) noted:

> Recent theoretical and empirical work have focused on the importance of the notion of unit in rational number concept aquisition. Novillis (1973) in a study of a hierarchy of rational number sub-concepts with 10- and 11-year-olds observed that these children had difficulty identifying the unit in region and number line representations of rational numbers. In a more detailed study of this phenomena, Novillis

> (1976) found that while 12-year-olds could successfully locate rationals less than one on a number line from zero to one, they had great difficulty doing so if the number line was extended. It is apparent from these results, as well as observations from Owen and Babcock in their work, that children at least up to the age of 12 or 13 have difficulty disassociating the notion of unit from "everything else". Thus a child, when given a picture of 4 cakes and aked to shade 1/2 a cake, tends to shade two cakes and a similar phenomenon exists with more frequency in number line settings. (p. 10)

In the first conference of the IGMPE at Utrecht, Streefland (1978) discussed some observational results concerning fraction concepts. He noted that

> the subdivision of continuous or dicrete quantities in equivalent parts is almost always the only way used to introduce fractions in primary education, ... [Moreover] after a few illustrations ... the equivalence of fractions is almost exclusively dealt with in an algorithmic way. (p. 51)

Freudenthal (1973) has similarly suggested that instruction has put a premature emphasis on learning algorithms.

Streefland (1978) cited the need

> of emphasizing the relationship between equivalent parts and quantities resulting from subdivision of an arbitrary unit and the fractional names, which enable us to describe those parts and quantities. (p.53)

Payne (1976) also emphasizes this point, and, as will be apparent in the next section, delineates it clearly in the Initial Fraction Sequence.

Organizing phenomena and situations which have something essential in common and using concrete models on the logical level of the child are also points confirmed by Streefland.

He concluded with these comments:

> The mental constitution of a concept is determined by the wealth of phenomena which discharge this concept, or by the ways education utilizes this wealth. As far as the concept of fraction is concerned, the didactical approach is based on a defective phenomenological analysis. As the observations showed its mental development involves many aspects. Besides, the different aspects of the concept are at different states of development, at a particular moment.
>
> It seems possible to identify situations in which a concept is concretized and which are so convincing to the child that they can serve as paradigms . . . .
>
> The material on which the mental constitution of the concept of fraction is based will then have a definitive meaning, even when the concept is tending to operate at an algorithmic level. (pp. 72-73)

It should be clearly evident from the citations and comments that there is much agreement among researchers as they consider both components of the process of learning and teaching fraction ideas, and as they consider needed research.

Instructional Sequences on Fractions

Some of the results of the research efforts at The University of Michigan were presented by Coxford and Ellerbruch (1975), in a chapter in the Thirty-seventh NCTM Yearbook. They noted that when the child comes to school, it is likely that the "part of something" idea is his or her concept of fraction. This concept must be extended, with attention focused on three questions:

(1) What is the unit ?
(2) How many pieces are in the unit ?
(3) Are the pieces the same size ?

They stressed that "much emphasis must be placed on equal size pieces", and that the child must be made aware that fractions are used to answer the question, "How much ?" The child must be able to make six connections, between the concrete model form, the oral form, and the written form. Children should have a great deal of experience working with concrete materials, which should be used to establish all six connections before using diagrams to represent regions or lengths, but delaying the use of sets of objects until the children's understanding of fractions used with regions and segments is very firm.

In the most recent NCTM Yearbook, Ellerbruch and Payne (1978) present in detail the Initial Fraction Sequence developed in the Michigan Studies. As the title of their article suggests, the teaching sequence provides a guide to instruction of (1) initial concepts of fractions, (2) the addition of like fractions, (3) equivalent fractions, and (4) the addition of unlike fractions. They begin by noting the importance of ascertaining prerequisite knowledge and teaching any lacking prerequisites:

> Before beginning computation with fractions, pupils must be able to represent fractions using concrete objects and diagrams, to recognize and use both oral and written words for fractions, and to recognize and use symbols for fractions. Consequently, this teaching sequence begins with a thorough introduction to fraction concepts. Next, the addition of like fractions is developed, using the single mathematical model of the measure of rectangular regions. The same model is then used to demonstrate that equivalent fractions name the same part of a given region. Rectangular regions are used to develop the generalization for expressing equivalent fractions in higher terms and then used in the addition of unlike fractions. (p. 129)

It is of importance that they based the introduction to fractions on a single mathematical model, a point both their experiences with children and their research con-

firmed.

The major steps in the sequence for developing initial fractions concepts, language, and symbols are then given:

The child must learn

1. to use concrete objects and make equal-size partitions;
2. to recognize and use the oral names for the various-size parts;
3. to draw diagrams of the concrete objects and attach the oral names to the parts;
4. to use the concrete objects and diagrams together with the oral names to write the fraction symbols. (p.131)

Of particular importance is the placement of fraction names and symbols:

> Research has indicated that pupils make a substantial number of "reversals when writing the numerator and denominator . . . When the oral names were taught before the written symbols, however, practically no reversal errors were made. This is why stress is placed on teaching the oral names before the written symbols. (p. 135)

Major steps in the sequence for generating equivalent fractions in higher terms are delineated:

The student must

1. recognize that <u>two equivalent fractions name the same amount</u>, or show the same measure;

2. develop and state the <u>generalization</u> that multiplying the numerator and the denominator by the same number generates an equivalent fraction;

3. find the "new"numerator, given a fraction and a "new" denominator;

4. when given a pair of fractions and a common denominator, find a "new" pair of fractions using the common denominator;

5. when given a pair of fractions, determine a common denominator and use this common denominator to find two equivalent fractions. (pp. 140-141)

Ellerbruch and Payne point out that

> The first two steps are easily related to the spatial models pupils learned earlier with rectangular regions. Steps 3, 4, and 5 are perceptually more difficult, at least when initially taught. Consequently, these last three steps are taught first using a set of rules and subsequently by relating them to the spatial models. Ellerbruch (1975) found that with this sequence pupils attained better skill with equivalent fractions and that their understanding was comparable. This is in contrast to a sequence where the spatial models are used to teach steps 3 through 5 prior to teaching rules. (In a similar study, Choate [1975] found that using concrete models simultaneously with rules for comparing fractions was not effective.) (p. 141)

It is to be expected that continuing efforts on the part of Payne and others will result in delineation of steps to extend the child's developmental of rational number ideas and skills with the operations.

Ratio and Proportionality

Ratio is a topic which has long been recognized as difficult by both mathematics and science teachers. Comparatively little research has been conducted specifically on ratio or the teaching of ratio concepts, however. Rather, researchers' attention has largely been turned toward proportionality, largely as a reaction to the work of Piaget, with numerous attempts to confirm or extend or refute his

hypotheses and findings. Inhelder and Piaget (1958) postulated that the learner does not have the cognitive skills necessary for an understanding of proportionality until the age of 12 or 13; ratio is a task requiring formal reasoning.

Piaget's theory proposes that four cognitive structures called logical operations (identify, negation, reciprocity, and correlation) unite to form a single structure, the INRC Group. The group composition then allows operations on operations, supporting the development of proportionality.

Piaget investigated the child's aquisition of proportionality by examining children's reactions to such situations as equilibrium on a balance and shadow size. He found that younger children (aged 7 to 12) dealt with these problems arithmetically. On enlargement tasks, they assumed that any enlargement is sufficient, or they decided on a fixed enlargement no matter what the question asked. Older children demonstrated understanding of proportional increase and decrease, as well as reciprocity between various relations. They overtly indicated the use of mathematical relationships between variables. Children demonstrated an intuitive understanding of proportionalty before they could deal with it quantitatively, however.

Affirmation of Piaget's results on proportion reasoning has come from studies by Lovell (1961) and Lunzer (1965) among many others; both concluded that only rarely have average to bright junior high school children attained the formal reasoning stage. In studies with older secondary school students (e.g., Lawson and Blake, 1976) or with college students (e.g., McKinnon, 1976), it has been found that at least half the students had not attained the formal level.

This is a larger percentage as expected, and the majority of those who attain formal reasoning attain it at a later age than Piaget predicted.

A few studies investigated the possibility of speeding up the transition from concrete to formal operations. The results of most of these studies have been less than successful. Lawson and Wollman (1976), for instance, reported specific transfer taking place, but not non-specific transfer. It may be that the subjects judged to be transitional made the most growth on tasks in which no training was given. This result seems somewhat consistent with the conservation training studies in which children's progress depended on their initial developmental level.

Successful results have been reported by several researchers. McKinnon (1970) found that a group which took an inquiry course (in science) made a significantly larger gain on the posttest than the group which took the regular science class. Fischbein et al. 1970) found that

> 9- to 10-year-old Ss, after a brief instruction, become able to perform chance estimates by comparing numerical ratios and to understand the concept of proportionality . . . (p. 388)

They also hypothesized:

> the data now available in genetic psychology with regard to the difficulty of accelerating the transition to a higher developmental stage refer mostly to the transition from the intuitive to the concrete operational stage. Would it not be reasonable to assume that during the process of mental development the differences between the successive stages are gradually reduced ? If so, instruction might be able to set up structures corresponding to formal operations already at the concrete operational stage with much greater ease and more stability than would be the case for the transition from the preoperational to the operational stage. (p. 388)

Karplus et al. (197oa, 1970b, 1972, 1974a, 1974b) have isolated various strategies that children use to explain their answers to a proportion problem. In a series of five studies, they classified responses from a group-administered, paper-and-pencil test according to a hierarchical schema, with categories ranging from concrete to abstract levels of thought. The studies were intended to indicate how well ratio concepts, in particular, function at various ages and to explore conditions under which learning these concepts might be facilitated.

The tasks included the Island Puzzle, two forms of the Paper Clips task, the Candy task, the Ruler task, the Pulley task, one geometrical task intended to assess ability to recognize a fraction of a whole, and one numerical task requiring the application of proportional reasoning. These were administered to groups from grade 4 through adult level.

Easley and Travers (1976) summarized the interpretation of the five studies by noting that all the five studies revealed a tendency for the median frequency of tasks to move from lower-level to higher-level categories as grade level increased.

> The researchers, overall, were disturbed by the implications of their findings. In study I, intellectual development, as assessed by their taxonomy, reached a "disappointingly low level" in the high school age group and did not progress much further. In study II, it was found that successful proportional reasoning was not reached until the last years in high school. This concern was reiterated in III, where evidence was found that many students did not advance to more abstract categories of thought during the intervening two years of that longitudinal study. Another disturbing implication, prompted by the data of studies II and III, was evidence of apparent obstacles to learning which may be inadvertently set up by "mathematics courses, by teachers, and by the chidren's cultural environment".

> Of particular notice was the dramatic contrast between the responses of urban and suburban 11th and 12th grade students. It was found that 80% of the suburban students, but only 9% of the urban students, were classified at the highest level P. (p. 39)

Karplus et al. (1974a) placed emphasis on the context in which the problem was presented. Form B of the Paper Clips Task was more abstract than Form A, forcing students to make use of the data. They concluded that perhaps students' attitude toward handling of data was reflected in their responses. It was also apparent in Wollman and Karplus (1974b) that tasks tending toward concreteness (Ruler and Paper Clips tasks) led to more correct responses than did the abstract tasks. Easley and Travers summarize:

> The lack of applicability of proportional reasoning to physical relationships raised questions about the appropriateness of many instructional strategies, particularly at the junior high school level, where only about 15% of the subjects were found to have reached the highest level. The researchers speculated that one source of the problem may be that ratios are introduced as fractions and proportions as equivalent fractions. (p. 39)

Another outcome of the analysis is the finding that younger children tended to make predictions by guessing on the basis of appearances and used the numbers they were given haphazardly. The children in the middle age range used two different kinds of strategies. They either used all the data by computing the difference between two of the measurements and adding the difference to the third measurement, or they used a scaling factor which was not related to the ratio inherent in the measurements. The older children tended to use either a combination of the latter two strategies or were able to solve the problem using a proportion.

In their critical commentary on the set of articles, Easley and Travers (1976) state:

> It should be clearly noted that Karplus and his colleagues are investigating a different problem from that studied by Piaget . . . . [They] are careful not to identify what they call proportional reasoning . . . with Piaget's Formal Operations. At least, they are quite open to the possibility that these may turn out to be different things. . . . [Thus] we have evidence from high school seniors (and even many adults) who do not employ "abstract logic" or "proportional reasoning" in these problems, but we are not entitled to infer that they have not achieved the stage of formal operations as Piaget defines it, even though Inhelder and Piaget state that proportional reasoning is only attained at the stage of formal operations. (pp. 39-40)

They also point out that Piaget used interviews, with a probing technique, while Karplus used paper-and-pencil tests, with interviews only to check the levels of performance attained from the written responses.

> This difference in purpose and procedure explains in part the difference in age distribution found between the two groups of studies. A second major contribution to these differences, which the research to date cannot isolate from the first, is the phenomenon Piaget calls <u>decalage</u> . . . Another difference is that Karplus et al. (study V) employ a concept of types of reasoning which depends on the external static form of arguments. This contrasts sharply with Piaget's interest in internal dynamic processes. (p. 41)

Another difficulty was cited by Nelson et al.(1969) who demonstrated that, in checking proportionality, as in so many other problems in cognitive development,

> verbal deficiencies or ambiguities in task presentation often mask the actual capacity of the child to perform in the required manner. . . . Once the in-

> structions were clear there was no difficulty in comparing two proportions and there was no interference from the perceptual cues that otherwise presumably controlled their responses. (p. 261)

In Karplus et al. (1977), proportional reasoning and control of variables were studied with samples of 13- to 18-year-olds in seven countries (Denmark, Sweden, Italy, United States, Austria, Germany, and Great Britain). They noted that

> Previous studies of these areas have helped to identify the gradual progress in adolescent reasoning for population samples in the United States, Great Britain, and Switzerland (Karplus & Petersen, 1970; Levine & Linn, 1977; Lovell & Butterworth, 1966; Lunzer & Pumfrey, 1966; Suarez, 1974; Wollman, 1975). (p. 411)

The task for assessing proportional reasoning was similar to the Paper Clips Task, Form B. The task for assessing control of variables had students separate the effects of sphere weight and release height on the dependent variable, indicating how far a target sphere was pushed by a sphere rolling down the plane. They found that, on the proportional reasoning task, about 1/4 of the students gave ratio responses, 1/6 used additive reasoning, and the remainder were divided between intuitive and transitional categories. They concluded that

> The effects of gender were not associated systematically with a particular form of school organization, nor with classes segregated by gender. Significant differences that did occur always favored the boys. Girls tended to use additive reasoning more commonly than did boys.
>
> Differences in achievement among countries were much smaller than differences among groups within a country. Austrian boys distinguished themselves on proportionality reasoning, but Austrian students as a whole did not perform comparably well on control of variables. In two countries, Denmark and Germany, there was a much lower frequency of additive reasoning than in other countries.

> Both socioeconomic status and selectivity of school affected student performance significantly, though the magnitude of the effect depended on the task and on the country. (p. 416)

In a recent study with older students, Parete (1978) developed a paper-and-pencil test to assess formal reasoning level, validating it with a group of community college freshmen. The test included four items on proportional reasoning (an enlarging machine problem, a salesman's commission problem, a sales tax problem, and a magnifying glass problem). He found that 23.4 percent of the students were at the early formal stage and 37.2 percent were at the late formal stage; the remainder were at early (22.9%) or late (16.5%) concrete stages. He noted that

> The distribution on the Proportionality Task seems to be shifted more toward advanced substages than are the distributions for the other task.This may be due to the fact that mathematics problems which require proportional reasoning for a solution are introduced quite early in the public school mathematics curriculum. Although Piaget has claimed many times that his evaluation of sujects' reasoning levels is content free, recently he had admitted that a subject's familiarity with the content of any of his tasks may in some degree influence the subject's performance on the task, especially in older subjects. (p. 41, manuscript chapter 4)

The addition strategy, the scaling strategy, and finally, consistently forming equivalent ratios were noted.

Thus, the results on many studies indicate the wide variance in the ages at which students can perform on proportional reasoning tasks using mathematical relationships. In fact, it may be that some learners never attain the formal reasoning level. This, obviously, has implications for instruction.

Learning and Instruction on Ratio and Proportion

In this section are cited studies which considered specifically factors related to learning and teaching ratio and proportion ideas in school programs. They emphasize some of the questions apparent previously.

Hart (1978) described aspects considered as basic to the development of ratio concepts by the Concepts in Secondary School Mathematics project in Great Britain:

(1) doubling or halving

(2) multiplication by an integer

(3) given a ratio per unit, apply this rate

(4) finding a rate per unit and then applying it

(5) enlarging a drawing in ratio 2:1, 3:2, 5:3, etc.

(6) finding a ratio a:b using an intermediate quantity, c

(7) using a fractional multiplier

(8) simple percentages

She reported that administration of the test to about 200 children indicated that

> Doubling ability is no indicator of the ability to tackle other ratio questions. Children have tortuous methods of obtaining larger numbers, the addition strategy being fairly consistent on harder questions. Techniques dealing with fractions or percentages are remembered incorrectly and indeed answers which are outside any sensible range are not recognized as such. This was summed up perhaps by one child who said "I was doing it mathematically not logically". (p. 6)

Steffe and Parr (1968) tried to partition out the potential cause of difficulty due to the mode of presentation of a

proportion problem. They reported that:

1. There is little correlation between the ability of children at the fourth, fifth, and sixth grades to perform successfully in proportionality situations at a symbolic level, such as $\frac{6}{15} = \frac{\square}{5}$, and their ability to perform sucessfully on porportionality situations based on ratio or fractional pictorial data.

2. Children solve many proportionalities presented to them in the form of pictorial data by visual inspection both in the case of ratio and fractional situations.

3. Whenever the pictorial data, which display the proportionalities, are not conductive to solution by visual inspection, the proportionalities become exceedingly difficult for fourth, fifth, and sixth grade children to solve, except for the high ability sixth graders.

4. For the denominator test, the proportionalities represented pictorially by a ratio situation were easier for the children to solve than the proportionalities represented pictorially by a fractional situation.

5. The children of high intelligence are much more adept at solving porportionalities for both a symbolic and pictoral representation than are children of low intelligence.

6. The fifth and sixth graders performed significantly better than the fouth graders on all tests and subtests involved. (p. 26)

As implications, Steffe and Parr noted:

1. Much more care must be taken in the fifth and sixth grades to develop a sequence of lessons which are designed to enhance children's ability to represent visual data mathematically in the case of ratio or fractions, indeed if that ability can be enhanced.

2. Special attention must be devoted to the lower ability children . . . (p. 26)

Other researchers have noted other attributes necessary in the development of ratio and proportion ideas. Abramowitz (1975b), for instance, noted that it is necessary to have the ability to make a distinction between "bigger than" and "times as much", as well as the ability to understand inverse relations between unit size and the number of units used in a measuring task. As she considered the results of previous research and her own study, she indicated that

> These results suggest that those investigating the developmental acquisition of proportionality must be careful not to generalize too quickly from performance on any one proportion task to the concept of proportionality in general.
>
> It seems possible for Ss to have an intuitive understanding of proportion without concurrently having the mathematical facility to solve proportion problems.
>
> A surprising result from the factor analysis is that skill tests of facility with fractions load on a different factor than tasks involving proportionality . . . . proportion tasks demand a knowledge of this facility but also an understanding of how to and when to use it in an appropriate situation . . . . It suggests that drill alone may be insufficient in teaching proportion. The teaching of fractions must be supplemented with tasks which help students conceptualize what they are doing with these numbers.

Karplus et al. (1977) believe that the findings from their study have implications for teachers:

> First of all it is clear that a substantial fraction of students between 13 and 15 years of age lack the ability to articulate proportional reasoning and/or control of variables. Science and mathematics programs in all but the top level of selective schools should take this diversity of student reasoning into account in so far as content selection, laboratory activities, and textbook choices are concerned.
>
> Second, it appears from the small but significant country-to-country differences that teaching can have some influence on the development of reasoning by the students in the age range being investigated here. Applying proportional or

> control-of-variables reasoning is not the result of a process exclusively internal to the young people.
>
> Third, the development of the reasong patterns considered in this paper should be an important objective of teaching programs for 13- to 15-year-olds, an objective that is not achieved comprehensively by present school practice in any of the countries included in our study. Other investigations (Kurtz, 1976; Lawson & Wollman, 1976) have identified certain promising teaching approaches that have been presented elsewhere (Karplus et al., 1977). A key component of these approaches is to direct the attention of students at their own reasoning, the thought processes by which they have arrived at valid or invalid conclusions, correct or incorrect answers. Other aspects are based on (1) identification of the reasoning required for the understanding of science concepts, (2) arranging for active participation in learning by the students, and (3) providing concrete experiences at the beginning of new topics (pp. 416-417)

Those who have developed teaching materials on ratio (e.g., Higgins, 1967; Geisert, 1972; Kurtz, 1977) have reported success. Furthermore, Ford (1974) found that pupils who had used a textbook emphasizing ratio and proportion in grades 5 and 6 scored higher in grades 6, 7, and 9 than those who had had a program in which proportion was not introduced until grade 7.

Further study of ratio and proportion ideas is obviously needed, with clearer delineation of what is entailed and how the ideas are acquired. Instructional sequences, such as those being developed for fractions, may be needed to help to fulfill the promise implied by the last-cited study.

SUMMARY

This paper presents highlights from several sources, in order to indicate the status of knowledge about fraction and ratio concepts and operations, to draw attention to comments from researchers about their findings, and to note points to consider as research proceeds.

Data indicate that many students have little conceptual understanding and computational skill with fractions; performance on fractions is far lower than with whole numbers. Errors students made in solving fraction items indicate how students manipulate numbers without understanding. Questions have arisen about the appropriate age level at which to provide instruction on fractions, since achievement results are so poor with the current placement.

- Researchers have focused not on postponement, but on sequencing and on clarification of the construct itself.
- Emphasis should be placed on providing a sound initial development of fraction concepts using concrete materials.
- The development of fraction ideas should be paced so as to connect firmly with the main points in the initial development.
- Instruction has put a premature emphasis on learning algorithms.

Seven interpretations on rational numbers have been proposed and discussed in depth in terms of implications for curriculum development and research. The fraction concept is complex, made up of a variety of components or subconcepts and attributes.

- Both short- and long-range objectives for instruction must be considered in relation to the interpretations of rational numbers. Then appropriate interpretations should be selected to develop certain objectives, necessary cognitive structures to meet the objectives ascertained, and sequences of instructional activities to contribute to the growth of these structures determined.

A comprehensive review of studies conducted at the University of Michigan includes such reflections as:

- It is important to do an analysis of the various components of the learning structure for an algorithm.
- It appears that algorithms were learned best when they were logical, and retained best when there was a heavy visual component to instruction.
- Evidently it is much more complicated to relate a child's thought to use of concrete materials or diagrams than is usually assumed.
- Word names should be taught before fraction symbols.
- The set model seemed to interfere with the measurement model.
- Relating a spatial representation and a verbal algorithm helps in learning equivalent fractions.
- It takes much longer to teach fractions than has generally been allocated in the curriculum or instructional materials.
- In the presentation of the Initial Fraction Sequence developed at Michigan, rectangular regions are used as the single mathematical model for the introductory work.

Comparatively little research has been conducted specifically on ratio concepts. Research attention has largely been turned toward proportionality, largely as a reaction to the

work of Piaget. It appears that no more than half the students have attained proportional reasoning even by age 18. Certainly there is wide variance in the ages at which students can perform on proportional reasoning tasks by using mathematical relationships. Additive and scaling strategies are used by many.

- Pictures aid children in solving proportionalities. Concrete experiences and active participation appear necessary.
- Much care must be taken to develop a sequence of lessons designed to enhance ability to represent data mathematically for ratios and fractions.
- A key component of instruction is to direct the attention of students to their own reasoning, to the thought processes by which they arrived at valid or invalid solutions.

It appears that instruction can influence the attainment of ratio and proportion ideas, just as it can influence the attainment of fractional ideas. Research and curriculum development must be synchronized to provide instructional programs appropriate for children's psychological and mathematical status and needs.

REFERENCES

Abramowitz, Susan. Proportionality: As Seen by Psychologists and Teachers. ERIC: ED 111 689. 1975 (a).

Abramowitz, Susan. Adolescent Understanding of Proportionality: Skill Necessary for Its Understanding. ERIC ED 111 690. 1975 (b).

Babcock, G. The Relationship Between Success with Linear and Area Measure Concepts and Achievement on Fractional Number Concepts. Research proposal, University of Alberta, 1976. Cited by Kieren, 1976 (b).

Bright, George. Assessing the Development of Computation Skills. In Developing Computational Skills (Marilyn N. Suydam, editor). 1978 Yearbook. Reston, Virginia: National Council of Teachers of Mathematics, 1978.

Carpenter, Thomas; Coburn, Terrence G.; Reys, Robert E.; and Wilson, James W. Results from the First Mathematics Assessment of the National Assessment of Educational Progress. Reston, Virginia: National Council of Teachers of Mathematics, 1978.

Coxford, Arthur and Ellerbruch, Lawrence. Fractional Numbers. In Mathematics Learning in Early Childhood (Joseph N. Payne, Editor). Thirty-seventh Yearbook. Reston, Virginia: National Council of Teachers of Mathematics, 1975.

Easley, John A. amd Travers, Kenneth J. Expanded Abstract and Analysis of "The Karplus Studies". Investigations in Mathematics Education 9: 34-42; Spring 1976.

Ellerbruch, Larry W. and Payne, Joseph N. A Teaching Sequence from Initial Fraction Concepts through the Addition of Unlike Fractions. In Developing Computational Skills (Marilyn N. Suydam, editor). 1978 Yearbook. Reston, Vir-

ginia: National Council of Teachers of Mathematics, 1978.

Fischbein, Efraim; Pampu, Ileana; and Manzat, Ion. Comparison of Ratios and the Chance Concept in Children. Child Development 41: 377-389; June 1970.

Ford, Elinor Rita. The Spectrum of Porportion Mastery Among New York Parochial School Children. (Columbia University, 1974.) Dissertation Abstracts International 35A: 3573; December 1974.

Freudenthal, H. Mathematics as An Educational Task. Dordrecht, Holland: D. Riedel Publishing Co., 1973.

Geisert, Pau George. A Study of the Hierarchical Competencies Underlying the Problem Solving Use of Proportion. (The Florida State University, 1971.) Dissertation Abstracts International 32A: 5103-5104; March 1972.

Gillet, Harry O. Placement of Arithmetic Topics. Journal of the National Education Association 20: 199-200; June 1931.

Ginther Joan; Ng, Katie; and Begle, E.G. A Survey of Student Achievement with Fractions. SMESG Working Paper No. 20. Stanford, California: Stanford Mathematics Education Study Group, October 1976. ERIC: ED 142 409.

Hart, Kathleen. The Understanding of Ratio in the Secondary School. Mathematics in School 7: 4-7; January 1978.

Hartung, Maurice L. Fractions and Related Symbolism in Elementary-School Instruction. Elementary School Journal 58: 377-384; April 1958.

Higgins, Jon Lyle. The Development and Evaluation of Mathematics Curriculum Materials for Use in a Junior High School Physical Science Program. (The University of Texas, 1967.) Dissertation Abstracts 28A: 1620; November 1967.

Inhelder, B. and Piaget, J. The Growth of Logical Thinking from Childhood to Adolescence. New York: Basic Books, 1958.

Karplus, Elizabeth F. and Karplus, Robert. Intellectual Development Beyond Elementary School I: Deductive Logic. School Science and Mathematics 70: 398-406; May 1970 (a).

Karplus, Robert and Peterson, Rita W. Intellectual Development Beyond Elementary School II: Ratio, A Survey. School Science and Mathematics 70: 813-820; December 1970 (b).

Karplus, Robert and Karplus, Elizabeth F. Intellectual Development Beyond Elementary School III: Ratio: A Longitudinal Study. School Science and Mathematics 72: 735-742; November 1972.

Karplus, Elizabeth F.; Karplus, Robert; and Wollman, Warren. Intellectual Development Beyond Elementary School IV: Ratio, the Influence of Cognitive Style. School Science and Mathematics 74: 476-482; October 1974 (a).

Karplus, 1974 (b): See Wollman and Karplus, 1974 (b).

Karplus, Robert; Karplus, Elizabeth; Formisano, Marina; and Paulsen, Albert-Christian. A Survey of Proportional Reasoning and Control of Variables in Seven Countries. Journal of Research in Science Teaching 14: 411-417; 1977.

Kieren, Thomas E. On the Mathematical, Cognitive, and Instructional Foundations of Rational Numbers. In Number and Measurement (Richard A. Lesh and David A. Bradbard, editors). Columbus, Ohio: ERIC/SMEAC, 1976 (a).

Kieren, Thomas E. Research on Rational Number Learning. Athens: Georgia Center for the Study of Learning and Teaching Mathematics, 1976 (b).

Kurtz, Barry Lloyd. A Study of Teaching for Proportional Reasoning. (University of California, Berkely, 1976.) Dissertation Abstracts International 38A: 680; August 1977.

Lankford, Francis G., Jr. Some Computational Strategies of Seventh Grade Pupils. Charlottesville, Virginia: The University of Virginia, October 1972. ERIC: ED 069 496

Lawson, Anton E. and Blake, A. J.D. Concrete and Formal Thinking in High School Biology Students as Measured by Three Separate Instruments. Journal of Research in Science Teaching 13: 227-235; 1976.

Lawson, Anton E. and Wollman, Warren T. Encouraging the Transition from Concrete to Formal Cognitive Functioning -- An Experiment. Journal of Research in Science Teaching 13: 413-430; 1976.

Lovell, Kenneth. A Follow-Up Study of Inhelder and Piaget's "The Growth of Logical Thinking". British Journal of Psychology 52: 1490; 1961.

Lunzer, Eric E. Problems of Formal Reasoning in Test Situations. In European Research in Cognitive Development (P.H. Mussen, editor). Volume 30, Monographs of the Society for Research in Child Development. Chicago: University of Chicago Press, 1965.

McKinnon, Joe W. The College Student and Formal Operations. In Research, Teaching and Learning with the Piaget Model (John W. Renner et al., editors). Norman, Oklahoma: University of Oklahoma Press, 1976.

Nelson, Katherine J.; Zelniker, Tamar; and Jeffrey, Wendell E. The Child's Concept of Proportionality: A Re-examination. Journal of Experimental Child Psychology 8: 256-262; October 1969.

Novillis, C. The Effects of the Length of the Number Line and the Number of Segments in Each Unit Segment on Seventh Grade Students' Ability to Locate Proper Fractions on the Number Line. 1976. Cited in Kieren, 1976 (b).

Owen, D. Pre-Measurement Abilities and Initial Fraction Learning in Children. 1976. Cited in Kieren, 1976 (b).

Parete, Jesse David, Jr. Formal Reasoning Abilities of College Age Students: An Investigation of the Concrete and Formal Reasoning Stages Formulated by Jean Piaget. Unpublished Docatoral Dissertation, The Ohio State University, 1978.

Payne, Joseph N. Review of Research on Fractions. In Number and Measurement (Richard A. Lesh and David A. Bradbard, editors). Columbus, Ohio: ERIC/SMEAC, 1976.

Piaget, J.; Inhelder, B.; and Szeminska, A. The Child's Conception of Geometry. New York: Basic Books, 1960.

Sambo, A.A. A Psycho-Mathematical Development of the Concept of Rational Numbers. A colloquium paper at the University of Alberta, Fall 1975. Cited in Kieren, 1976 (b).

Sambo, A.A. Transfer Effects of Measure Concepts on Fractional Number Learning. Research Proposal, University of Alberta, 1976. Cited in Kieren, 1976 (b).

Steffe, Leslie P. and Parr, Robert B. The Development of the Concepts of Ratio and Fraction in the Fourth, Fifth, and Sixth Years of the Elementary School. Technical Report No. 49. Madison: Wisconsin Research and Development Center for Cognitive Learning, March 1968.

Streefland, Leen. Some Observational Results Concerning the Mental Constitution of the Concept of Fraction. Educational Studies in Mathematics 9: 51-73; 1978.

Suydam, Marilyn N. and Dessart, Donald J. Classroom Ideas from Research on Computational Skills. Reston, Virginia: National Council of Teachers of Mathematics, 1976.

Suydam, Marilyn N. and Osborne, Alan. The Status of Pre-College Science, Mathematics, and Social Science Education: 1955-1975. Volume II: Mathematics Education. Final Report, National Science Foundation. Columbus, Ohio: The Ohio State University, 1977. Available from ERIC/SMEAC.

Suydam, Marilyn N. and Weaver, J. Fred. Using Research: A Key to Elementary School Mathematics. Columbus, Ohio: ERIC/SMEAC, 1975.

Wagner, Sigrid. Conservation of Equation and Function and Its Relationship to Formal Operational Thought. 1976. ERIC: ED 141 117.

Washburne, Carleton. Mental Age and the Arithmetic Curriculum: A Summary of the Committee of Seven Grade Placement Investigations to Date. Journal of Educational Research 23: 210-231; March 1931.

Wollman, Warren and Karplus, Robert. Intellectual Development Beyond Elementary School V: Using Ratio in Differing Tasks. School Science and Mathematics 74: 593-613; November 1974 (b).

SOME TRENDS IN RESEARCH AND THE ACQUISITION AND USE OF SPACE AND GEOMETRY CONCEPTS

Richard Lesh

This paper consists of two parts. The first part outlines the results of a multi-university research program dealing with space and geometry concepts which began in 1975. Papers from the research conference which initiated this effort are published in a monograph, Space and Geometry: Papers from a Research Workshop, edited by Martin (1977). A second monograph, Recent Cooperative Research Concerning the Acquisition of Spatial and Geometric Concepts, (Lesh and Mierkiewicz, 1978) contains fourteen research reports from projects associated with the above research effort. This later monograph will be available for the first time in September, 1978 - just in time for the Osnabrück Conference.

The second part of the paper will describe a major new cooperative research program that grew out of the space and geometry research described above. A research conference to initiate this new effort was held at Northwestern University on January 5 - 7, 1978. Papers from this conference will appear in monograph titled, Applied Mathematical Problem Solving (Lesh, Kantowski and Mierkiewicz). This monograph is scheduled to go to the printers in September, 1978. All three of the above monographs are available from the ERIC center at Ohio State University.

PART I: A COOPERATIVE RESEARCH PROJECT ON SPACE AND GEOMETRY CONCEPTS

In the spring of 1975, the Georgia Center for the Study of Learning and Teaching in Mathematics (GCSLTM) sponsored a series of five research workshops involving (a) teaching strategies in mathematics, (b) number and measurement concepts, (c) space and geometry concepts, (d) models for learning mathematical concepts, and (e) problem solving. This first part of this paper focuses on the activities of the working group that developed from the space and geometry workshop.

Some Assumptions Underlying the Work of the Space and Geometry Group:

The existence of a successful, nonfunded, multi-institutional, cooperative research effort is in itself a significant research innovation in mathematics education. Because of the complexity of most of the important issues in mathematics education, most issues will require intensive study by various individuals and long-term commitments and coordinated research efforts from groups of people. Furthermore, the optimum time to establish connections among individual research efforts is while project plans are in the formative stages - not a year or two after projects have been completed and the reports are appearing in journals or at conferences. When research projects are in the planning stages, it is helpful for groups of individuals working together to (a) identify basic issues that are important to as many people as possible, (b) formulate answerable questions related to the basic issues, (c) coordinate individual

projects so simplistic conclusions can be avoided, and (d) establish better bases of communication so individuals can profit by (and build on) the work of others. If research is restricted to answering questions through isolated studies, then asking important questions seems almost irreconcilable with asking an answerable question; and if the language and underlying theoretical constructs are not consistent across studies, the results are difficult to interpret or use.

Papers presented at the 1975 Space and Geometry Workshop

To review a broad survey of past research, to develop a language for dealing with these problems, and to describe some of the most important directions for future research, the following papers were presented at the 1975 space and geometry workshop.

Mathematical Foundations of the Development of Spatial and Geometrical Concepts ..... Edith Robinson

Piaget's Thinking about the Development of Space Concepts and Geometry ..... Charles D. Smock

Breakthroughs about the Development of Space Concepts and Geometry ..... Izaak Wirszup

Recent Research on the Child's Conception of Space and Geometry in Geneva: Research Work on Spatial Concepts at the International Center for Genetic Epistemology ..... Jacques Montangero

Needed Research on Space in the Context of the Geneva Group ..... Jacques Montangero and Charles D.Smock

Cross-Cultural Research on Concepts of Space and Geometry ..... Michael C. Mitchelmore

Transformation Geometry in Elementary School: Some Research Issues ..... Richard Lesh

The Proceedings of the space and geometry workshop were published in a book, Space and Geometry: Papers from a Research Workshop (Martin, 1976). In addition, several other Georgia workshops included papers pertaining to the acquisition of spatial and geometric concepts. For example, the "models" workshop (Osborne, 1976) included Martin's article, "The Erlanger Program as a Model of the Child's Conception of Space", and the "number and meaurement" workshop (Lesh, 1976) included several papers emphasizing the close links between the development of spatial/geometric concepts and the development of nature/measurement concepts.

From the beginning, the space and geometry group considered geometry to be a context to investigate general questions about concept acquisition. That is, research concerning the development of spatial concepts should not be an area of interest only to those who want to teach geometry. For example, many of the models and diagrams teachers use to introduce arithmetic and number concepts presuppose an understanding of certain spatial/geometric concepts. Consequently, misunderstandings about number concepts are often closely linked to misunderstandings about the models used to illustrate them. For this reason, one of the long-range goals of the space and geometry working group was to furnish information to help educators devise "better" instructional models for teaching basic arithmetic and number concepts - and to study some of the "modeling" processes students use to

link abstract mathematical concepts to real world phenomena.

After initial GCSLTM workshops, the space and geometry group maintained communication through a series of meetings at Northwestern University, at the University of Georgia, and at professional meetings. As a result of these planning sessions, the following set of 14 coordinated research projects were completed and published as the monograph, Recent Cooperative Research Concerning the Acquisition of Spatial and Geometric Concepts (Lesh and Mierkiewicz, 1978). The following table of contents gives a fairly clear idea about the major themes considered in these 14 studies. A description of these themes will be given in the next section of this paper.

"I. Studies Concerning Pre-Operational Concepts

Perception, Imagery, and Conception
R. Lesh & D. Mierkiewicz
Northwestern University

Apparent Memory Improvement Over Time?
R. Lesh
Northwestern University

Haptic Perception, Construction, and Drawing of Geometric Shapes by Children Aged Two to Five
K. Fuson & C. Murray
Northwestern University

The Role of Motor Activity in Young Children's Understanding of Spatial Concepts
J. Musick
Northwestern University

Mathematical Foundations for the Development of Spatial Concepts in Children

I. Weinzweig
University of Illinois
Circle Campus

II. Studies Concerning Transitional Stages from Concrete to Formal Operations

(a) Studies about Rigid Motions

Understanding of Selected Transformation Geometry Concepts

D. Thomas
Ohio State University

Variables Influencing the Difficulty of Rigid Transformations

K. Schultz
Georgia State University

Conservation of Length: A Function of the Mental Operation Involved

R. Kidder
Longwood College

An Investigation into the Effect of Instruction on the Acquisition of Transformation Geometry Concepts in First Grade Children and Subsequent Transfer to General Spatial Ability

F. Perham
University of Illinois
Circle Campus

(b) Studies about the "Middle" Geometries: Affine Transformations, Similarities, and Projections

An Analysis of Research Needs in Projective, Affine, and Similarity Geometries, Including an Evaluation of Piaget's Results

K. Fuson
Northwestern University

The Child's Conception of Rational Distances - An Affine Invariant

L. Martin
Missouri Southern State College

III. Studies Concerning Older Subjects or Formal Operational Concepts

Cognitive Studies Using Euclidean Transformations

J. Moyer
Marquette University, and
H. Johnson
Syracuse University

Understanding of Frame of Reference in Preservice Teacher Education Students

C. Dietz & J. Barnett
Northern Illinois University

Visual Influences of Figure Orientation on Concept Formation in Geometry

N. Fisher
Research Department
Chicago Board of Education

IV. An Analysis of Past Efforts and Direction for Future Research

A. Coxford
University of Michigan

Issues Considered by the Space Geometry Group

The following issues were central themes in the 1978 research monograph:

1. What general principles can be found for anticipating the relative difficulty of mathematical ideas? For example, if a child is operational (in the Piagetian sense) with regard to one concept, what does this imply about the child's ability to learn other "related" ideas? Van Hiele, Freudenthal, and several Soviet psychologists (e.g. Yakimanskaya, 1970 and Sergeevich, 1971) have described the way they believe geometric concepts evolve in children. Yet basic controversies and ambiguities occur in each of these theoretical descriptions; and the controversies strike at the heart of many of the most basic issues in develpmental psychology. If it is possible to find techniques to anticipate the relative difficulty of geometric concepts, then similar techniques may be useful to organize the sequential presentation of arithmetic and number ideas - or instructional models leading to arithmetic and number concepts.

2. Even within the category of "concrete materials", some materials are more concrete than others. Nonetheless, little has been done to investigate how the figurative content of a problem affects the difficulty of the task. Piaget has focused on the operational aspects of tasks and concepts and has deemphasized the figurative aspects. The influence of figurative content on operational ability is important information for teachers who must devise models to illustrate mathematical concepts.

3. One of the ingenious aspects of Piaget's theory is that he explicitly confronts the fact that ideas (as well as children) develop. That is, (a) a given idea can exist at many different levels of sophistication, (b) this evolution can be traced, and (c) the more primitive levels of a concept have seldom been investigated or accurately described. Yet, little is known about the nature of children's early conceptions of many mathematical ideas.

The first mathematical judgements children learn to make are highly specialized, closely tied to specific content, and involve highly restricted, "messy" primitive concepts that do not give rise to neat, tidy, elegant theory. For this reason, mathematicians have not taken the trouble to describe most of the structures children use when they first come to master a given idea. Nonetheless, to develop effective instructional materials, it is important to present ideas in a form that will be most understandable to children. Therefore, it is useful to formulate acceptable mathematical descriptions of children's primitive conceptions of important mathematical ideas.

4. Some of the best resources for describing the nature of children's early number concepts have come from Piagetian studies. Nonetheless, because Piagetian research has focused on the cognitive processes used by first-graders (i.e., INCR groups), children at intermediate levels of development have been neglected. Furthermore, because psychologists in general (and Piagetian psychologists in particular) have avoided mathematical ideas that are typically taught in elementary school, it is usually possible to make only relatively crude inferences about how children's mathematical

thinking gradually changes from concrete operational concepts to formal operational concepts. It is time for mathematics educators to apply Piagetian techniques and theory to concepts that exist at intermediate levels of development - as well as at adult or preschool levels.

5. In the past, the space and geometry group focused on "Piagetian" operational analyses of various concepts and on analyses of the figurative contexts in which these concepts were used. However, the availability of certain general problem solving strategies, as well as certain affective characteristics, also seem to relate to the solvability of a problem. Future research efforts may tend to focus more heavily on basic types of applied problem solving processes - that is, on processes students use to apply mathematical ideas to real world situations. These processes will be described in the second part of this paper.

Several Lines of Research Not Represented in the 1978 Monograph:

Several lines of research were considered to be important by members of the space and geometry working group - but were not represented in the 1978 monograph. Examples of such research areas include the following:

1. Computer simulation studies developing out of the work of Newell and Simon (e.g. Hayes & Simon, 1974; Newell & Simon, 1972). One of the best examples of research in this area is represented by the work of Greeno and some of his associates (e.g. Greeno, 1976 a & b).

Computers have been used to model the geometric construction and problem solving procedures used by students. In this way it has been possible to demonstrate that a given set of procedures is internally consistent and sufficient to perform a specified domain of tasks. Computer simulation research is of course preceded by a great deal of clinical observation and experimentation with students in real problem situations, and it has served to greatly extend and refine traditional "information processing" models of human learning. Current work in this area includes (a) the representation for visual imagery of abstract concepts (e.g. Schwarts, 1977; Kuipers, 1978) and (b) the influence of attentional demands and memory load on task performance capabilities (e.g. Pascual-Leone, 1976; Case, 1972).

2. Concept development studies based on the work of the Van Hieles.

Two studies in this area were attempted by various members of the space and geometry group. But, prohibitive problems arose in obtaining manuscripts of the Van Hieles' work. Neither of the studies was completed in spite of the fact that several interesting and well designed studies were considered. It is to be hoped that the kind of international cooperation this present conference is intended to encourage should make this a ripe area of future research.

3. Problem solving studies which focus on geometric content.

Examples of current work in this area include (a) studies based on the work of Krutetskii and other Soviet theorists (e.g. Kantowski and Landa), (b) studies based on

the work of Polya, (e.g. Schoenfeld), (c) studies focusing on "modeling" processes and applied problem solving processes (e.g. Lesh). References to this work will be given in the second part of this paper.

PART II: RESEARCH ON APPLIED PROBLEM SOLVING PROCESSES

The original space and geometry research group has now divided into two overlapping research groups - one focusing on the development of space and geometry concepts and the other focusing on applied problem solving processes.

As the first part of this paper suggested, the distinctions were not always clear between research aimed at studying (a) the development of spatial and geometric concepts, (b) "embodiments" or "concrete models" for illustrating arithmetic and number concepts, or (c) problem solving processes. For example, Kieren's (1978) recent work analyzing various mathematical interpretations (and corresponding concrete models) for rational numbers is closely related to research on "modeling" processes in space and geometry. Or, Carpenter's (1976) work with measurement concepts deals with figural models of basic number ideas. Similarly, regarding problem solving processes, a number of space and geometry studies were unavoidably forced to confront the issue of what it is, beyond <u>having</u> a concept, that allows a normally intelligent person to <u>use</u> the idea to deal with math-related problems in everyday situations. Piagetian studies have shown that the operational structure of an idea is an important factor determining the difficulty of the idea.

Geometry research has shown that the figurative content of the task situation can also radically influence the difficulty of a given problem situation. And, a student's proficiency with certain problem solving processes also influenced the difficulty of tasks. However, the relevant processes do not necessarily correspond to the typical kinds of problem solving processes discussed by Polya, Krutetskii and others.

To focus on the kinds of processes that seemed to be involved in earlier space and geometry studies, and to review and synthesize relevant research from a variety of related research perspectives, a conference on "Applied Problem Solving" was held at Northwestern University on January 5 - 7, 1978. Papers from this conference are currently being edited and should be available from the Ohio State ERIC center by November 1978. A tentative list of papers and authors includes:

1. "Problem Soving" Versus "Applied Problem Solving": What is the Distinction? Lesh, Northwestern University.

2. A Survey of Several Psychological Perspectives: What Kind of Problems and Processes Do They Consider? Bell, University of Chicago.

3. Problem Solving Projects or Mathematical Abilities Projects Affiliated with the Georgia Center for Learning and Teaching Mathematics. Hatfield, University of Georgia.

4. Soviet-Style Research on Problem Solving. Kantowski, University of Florida.

5. Polya-Based Research on Problem Solving: Some New Directions. Schoenfeld, University of California - Berkeley.

6. Social/Affective Factors Influencing Problem Solving Ability. Higgins and Trimble, Ohio State University.

7. Computer Simulation Research on Problem Solving. Heller and Greeno, University of Pittsburgh.

8. Mathematical Learning Disabilities. Lesh and Jones, Northwestern University.

9. Modeling Processes. Kieren, University of Alberta.

10. Information Processing Models of Problem Solving. Carpenter, University of Wisconsin.

11. Routine Problem Solving. Sowder, Barnett, and Vos, University of Northern Illinois.

12. An Analysis of Problem Solving Research Methodology. Nelson, University of Alberta.

In the remainder of this paper, I would like to identify some themes that I believe will be important in future research efforts on applied problem solving processes.

What is Applied Problem Solving?

What is it, beyond having an idea, that allows a normally intelligent person to use the idea to deal with math-related problems in everyday situations. The upcoming monograph is an attempt to clarify a productive scope and focus for research on applied problem solving - as

well as to identify an agenda of priority research issues. Because the papers for this monograph are still in the final formative stages, it would be premature to summarize their conclusions in this paper. However, the following sections taken from my own papers may serve to indicate some of my biases. I hope they will serve as points to stimulate discussion at the Osnabrück Conference.

1. Focus on Processes Needed to Use Mathematical Ideas:

In general, "being able to use a concept" involves something more than simply "having the concept". Getting an idea into a student's head does not guarantee the new idea will be integrated with other ideas that are already understood, nor does it guarantee the student will be able to use the library-type "look-up" skills that allow the idea to be retrieved when it is needed. "Being able to use an idea" may also involve problem solving processes that are not necessary in order to demonstrate the simple attainment of a concept. But what are these processes? Past research on problem solving seems to have made little progress. Perhaps the field is in need of reanalysis. Some promising new directions are described below.

2. Focus on Average Ability Students:

Most information about problem solving processes has come from situations involving older students, exceptionally bright students, individual students working in isolation (often in artificial laboratory situations), or situations involving highly contrived word problems,

mathematical puzzles, or proofs. Elementary school children, average (or below average) ability students, and applied problem solving processes have been neglected. For this reason, "the problem solving" processes educators discuss often seem inaccessible to younger children or less gifted students, and applied problem solving processes, like modeling, have been ignored.

Traditionally, a great deal of the research in problem solving has focused on identifying and analyzing the abilities and processes used by bright students and good problem solvers. The assumption is that average and below average ability youngsters can be taught to use the processes used by good problem solvers. Krutetskii (1966), however, has shown that this assumption may be unwarrented. That is, many of the abilities and processes used by gifted youngsters may be inaccessible to average ability youngsters. If this conclusion is correct, another fruitful tack for research to take may be to analyze the disabilities of poor problem solvers or the difficulty-causing processes "learning disabilities" students - and to investigate the extent to which these processes and disabilities also cause difficulties for average ability problem solvers. Many assumptions about learning and problem solving that seem quite sensible when they are used to describe average ability youngsters become obviously absurd when they are applied to exceptional children - either "gifted" or "learning disabled".

3. Focus on Processes Related to Concept Formation:

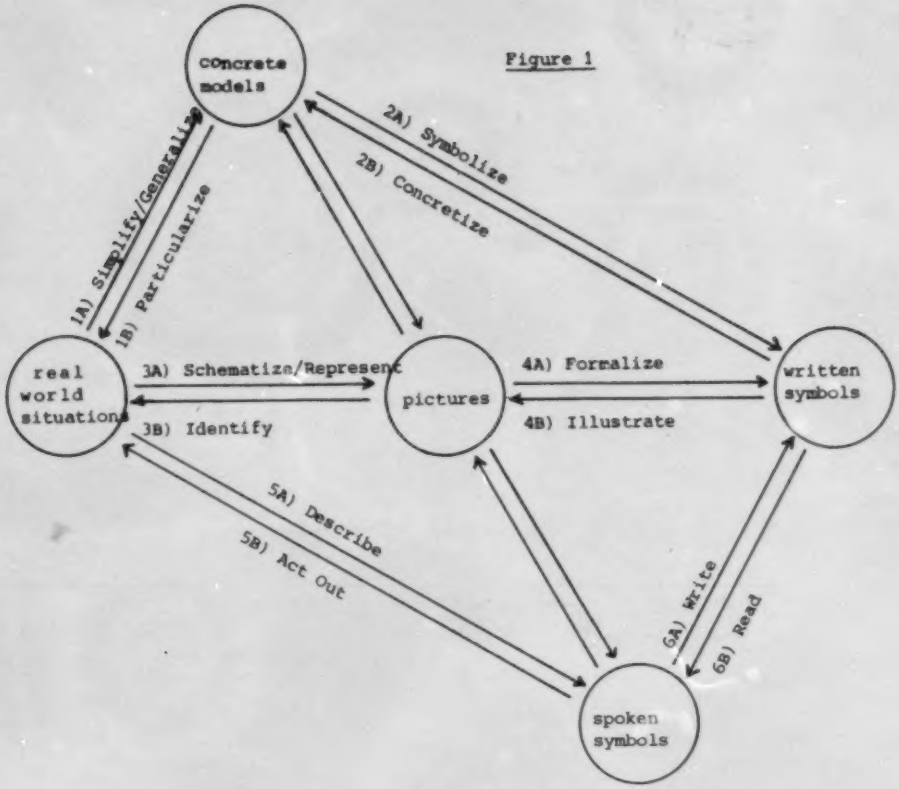
The definition of "problems" must take into account the prior experience cognitive capabilities of the student. A problem for one student may be an exercise - or a bore - for another. But, more than this, some of the most

important problem solving processes probably also serve to increase the meaningfulness of available concepts. If applications are only considered to be appropriate after learning has occurred, then they will (and perhaps should) be neglected. If applications are to find a place in the curriculum, they must play an important role on the way to helping students understand the important ideas we want them to learn.

It seems questionable to claim that a person can apply a concept before the concept has been learned, but it seems equally questionable to claim that a concept is known if it cannot be used in simple applications. Perhaps, applied problem solving processes play an important part in the formation of most mathematical concepts. Or, perhaps it is naive to think of ideas as being either understood completely or else not understood at all. Perhaps it is more accurate to think of ideas as gradually becoming more and more meaning as they become more complex and are connected to more and more other ideas and events.

There is a variety of ways to make an idea meaningful - some of which involve "translation" processes corresponding to the arrows in Figure 1. These processes not only play an important role in the development of mathematical concepts, they are also among the most important "modeling" processes students use when they try to apply the concepts in real life situations.

Figure 1


concrete models
2A) Symbolize
2B) Concretize
1A) Simplify/Generalize
1B) Particularize
real world situations
3A) Schematize/Represent
3B) Identify
pictures
4A) Formalize
4B) Illustrate
written symbols
5A) Describe
5B) Act Out
6A) Write
6B) Read
spoken symbols

4. Focus on Modeling Processes

Using mathematics to solve real problems usually involves:

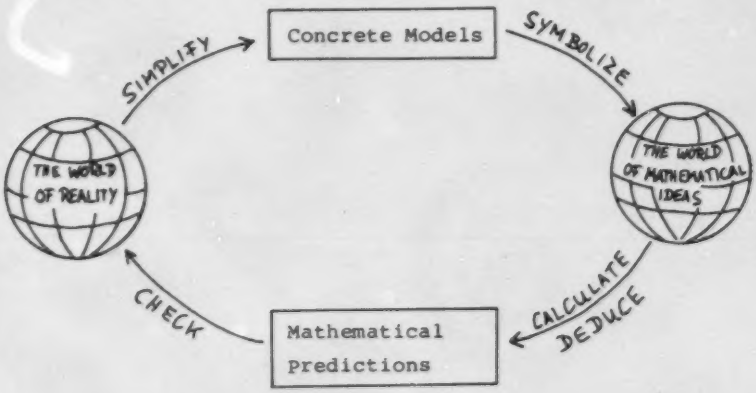
a) a dive from the world of reality into the world of mathematics;

b) a swim in the world of mathematics;

c) a climb from the world of mathematics back into the world of reality, carrying a prediction in our teeth.

But, the "dive" often requires us to ignore some things about the real situation in order to focus on others. That is, a series of successive simplifications are often needed before a situation "fits" a nice mathematical description.

Figure 2.



The processes in Figures 1 and 2 are important for a variety of reasons.

a) They are simplified versions of the "modeling" processes used by gifted applied problems solvers. They represent some of the most important processes students need when they try to use basic geometric, algebraic, or number concepts.

b) When we say a student "understands" a mathematical concept, part of what we mean is that he/she can use the kind of processes listed in Figures 1 and 2. Yet, students are given few instructional activities that focus directly on these processes - in spite of the fact they are the kind of processes that give meaning to the ideas teachers are trying to teach.

c) Average or below average students can learn to use these processes. Yet, work with special education students indicates these problems cause difficulties for many students - and that these difficulties can severely restrict problem solving (or even concept formation) capabilities.

d) Teachers do not need to wait for large scale curriculum projects to develop special instructional activities to teach these processes. They can be built into the kind of lessons that are already included in many textbooks and the kind of problems that are included in the "applied" sections of national assessment tests.

e) If diagnostic questions indicate a student is having unusual difficulties with one of the processes in Figure 1, other processes in the diagram can be used to strengthen or bypass the difficulty.

5. Focus on Translation/Representation/Notation Difficulties:

If space permitted, several other important "translation" processes could also have been discussed in connection with Figure 1. These include:

a) translating from one type of model to another,

b) translating from one type of picture to another,

c) translating from one verbal description to another, and

d) translating from one written statement to another.

One handicap with the above translation processes is that they seem rather insignificant compared with the more grandiose processes problem solving buffs usually like to discuss - e.g. reason by induction, reason by analogy, consider an auxiliary problem, or work backwards. On the other hand, it should be clear that the translation processes are really only simplified versions of other familiar problem solving processes - e.g. introduce suitable notation, look for a similar problem, simplify the problem, or restate the problem in your own words.

The more grandiose processes may be quite appropriate for older students or for exceptionally bright youngsters, but Krutetskii's work suggests they are often inaccessible to average or below average students.

6. Focus on Processes Needed to Use "Social/Technological" Amplifiers:

When educators talk about problem solving situations, they often ignore the fact that most people work on real world problems when other people and other resources are available. People seldom work in isolation using only the power of their own minds to solve problems. Instead, good problem solvers learn to amplify their own powers through effective use of outside resources. For example, when real people solve real problems, one of the most often used problem solving strategies is to "ask someone who can give the needed information". This is not to say that good problem solvers solve most problems by asking someone else to do their work for them. Formulating a problem in such a way that a specific bit of information can be requested is not a trivial skill. In fact, one of the most obvious characteristics of good problem solvers is that they are good question askers. Once a question is formulated in a nice way, answer giving is often quite easy.

Applied problem solving usually involves:

a) formulating the problem into an answerable question,

b) devising an appropriate procedure for getting the needed information,

c) carrying out the procedure, and

d) interpreting and evaluating the answer; and "answer giving is only involved at stage "c".

When a psychologist gathers information about students solving artificial problems in isolated laboratory situations, the data are usually analyzed using a computer. The psychologist does not use the computer because he is too ignorant to calculate the relevant statistics with a pencil and paper; he uses the computer because the important aspects of his own problem solving situation are to select and interpret the appropriate calculations - not to actually go through the drudgery of computation. Furthermore, to decide upon the most appropriate computations to make, the psychologist probably also consulted a statistician before he carried out his analysis. Again, this does not mean that the psychologist is ignorant about statistics; to formulate the problem in such a way that a statistician can help requires "translation" skills like the ones in Figure 1, and a firm understanding of statistics.

7. Focus on Information Retrieval Processes:

After a problem is interpreted in a meaningful way, relevant information must be retrieved from memory or from available resources. Often these "library-type" retrieval skills represent important aspects related to the meaningfulness of the ideas related to a problem. That is, as an idea is related to other ideas, its meaningfulness increases. Yet these processes of forming relationships among ideas are seldom taught either as concept formation skills or as problem solving skills. For example, even in the most obvious situations - say length concepts and area concepts - preestablished concepts are seldom related to new ideas being taught, and problem solving situations are seldom designed to help students form related networks of ideas.

8. Focus on Individual Stages in the Overall Problem Solving Processes:

Many people believe that the best way to learn to solve problems is simply by solving problems - and, afterwards, looking back to reconsider processes that were most helpful. Unfortunately, if the problem solving process is not broken into stages, and if students are forced to work entire problems in isolation, the "looking back" technique seldom works. Average problem solvers seldom solve problems quickly enough to look back on more than a few problems. So, the strategies they notice are often useful for only a small number of problems of a particular type, and they do not learn cues to determine when each strategy might be useful. Furthermore, average problem solvers often have difficulty focusing on processes they use to solve problems. Even when they are able to give correct answers, they are notoriously unable to give accurate descriptions of steps in the process they used.

To practice these skills, a teacher can give students a whole list of problems and ask them to postpone trying to solve any of the problems until they have first gone through the entire list - trying to determine which technique would be "best" for each problem. For example, for computational problems, students can "set up" each problem before attempting to actually carry out the procedures they selected; or the teacher can simply hand out the requested information. Finally, the group can work together to estimate the reasonableness of the information they receive. Schoenfeld has shown that learning a system of strategies or heuristics involves a good

deal more than simply learning each of the individual processes in isolation. A "managerial strategy" for selecting and evaluating processes is also needed.

By working in a group, students can investigate critical parts of more problems, and by working on whole sets of problems, one stage at a time, they can focus on "non-answer giving" stages in problem solving. Furthermore, by working with other students, average problem solvers often find it easier to focus on processes that would otherwise have been internal and more difficult to notice.

By focusing almost exclusively on the answer-giving stage of problem solving, and by treating problem solving as though it is always necessary to do all stages of the problem without any kind of assistance, students are often misled about what most real problem solving is really like. Furthermore, when problem solving is carried on in isolation, many useful problem solving strategies become almost impossible to teach to average or below average students. Some examples of these useful but difficult to teach strategies will be given in the next section of this paper.

9. Focus on "Group" Problem Solving Processes:

Many individual problem solving strategies are quite difficult for average or below average ability youngsters. But, when these internal processes are externalized in the context of small group activities, they are often easier to describe in a form that is understandable to lower ability problem solvers. For example, problem

solving strategies like "consider a similar problem", "consider an auxiliary problem", or "consider a special case", can be summarized with the simple advice, "look for a related problem". Yet, to poor problem solvers, this advice often seems quite foolish because, "I already have one problem I cannot do, I do not need another". To poor problem solvers, a more sensible suggestion is, "Look at the same problem from a different point of view".

In several recent research studies, where groups of four students were supposed to work together on problems, individuals often worked independently - each conceptualizing the problem in quite different ways, and each unconscious of misleading biases inherent in their own point of view. This is one reason why "brainstorming" is often a useful problem solving technique.

In group, "brainstorming sessions", students can be bombarded with a variety of different ideas and approaches, and can simultaneously become more self-critical about their own points of view. They can also be made to notice: (a) some people are good talkers while others are good listeners, (b) some people are good generalizers while others are better at working out details, and in general (c) a variety of different roles are beneficial to good problem solving. Good problem solvers must be flexible enough to switch quickly from one role to another while solving a problem.

Many other problem solving strategies are greatly simplified in group situations. Problem solving strategies like "identify the givens", "identify the unknowns", and "eliminate irrelevant information" all having to do with the general recommendation, "understand the problem".

However, this advice again seems rather useless to poor problem solvers whose superficial understanding of the problem often leads to selecting or eliminating information on rather artificial bases. On the other hand, except for specific recommendations about identifying knowns and unknowns, it is difficult for poor problem solvers to understand what it means to "understand the problem". It is much easier to say, "Use your own words to describe the problem to a friend" or "Describe some other problems like this problem". Poor problem solvers sometimes flounder with a problem for a long time before noticing (if asked) that they are unable to give a clear description of the problem to a friend. So, once again, group activities can force students to "understand the problem" and "organize the information given". Eventually, they may become self-critical enough to work on problems and no longer need group work to overcome their subjectivity and egocentrism.

When students work in groups to solve problems, they often see that many problems can be solved in a variety of ways - some of which are "better" than others. In fact, in research with gifted youngsters (e.g. Krutetskii, 1976), the hallmark of outstanding problem solvers is not so much whether answers are right or wrong but whether "clever" procedures were used. "Good" problem solvers are flexible thinkers who are capable of solving problems in several different ways; so, when one path is blocked, another route can be taken.

Speed alone is not always an accurate measure of problem solving ability. For instance, because of the impulsiveness and inflexibility of their thinking "learning disabilities" subjects may actually solve problems (if they

are able to solve them at all) faster than their normal peers. However, impulsivity and inflexibility are hindrances to good problem solving that can be compensated in group problem solving sessions.

The above points are not intended to imply that we should explicitly teach group problem solving techniques. Rather, group problem solving situations furnish an effective context to teach individual problem solving procedures - especially when the problem solver is at a relatively primitive level of skill acquisition. In addition to previously mentioned cognitive justifications for small group activities, a number of affective justifications are also apparent.

10. Focus on Social/Affective Factors:

In modern psychology it has become more and more clear that cognitive adaptation exists in an ecological system with other adaptation-seeking mechnisms, and is influenced by them. Learning, socializing, and adjusting cannot be completely separated. Much that we now call learning is social learning.

Many aspects of human learning that we have traditionally regarded as "cognitive" development are specializations of "social" development. The central issues are not so much about how the child develops knowledge, but rather, about how he develops shared or cooperative knowledge.

If we look at the moment-to-moment activities of children, we never find learning as a separate or isolated human activity. We have begun thinking about dialectical pro-

cesses, about sharings of reality among people that create not only the objective (i.e. intersubjective) reality of science, but also the individual's conception of the self (Mead 1934) and his moral ideology (Kohlberg 1963, 1969). Observed changes in cognitive performance are first and foremost, assimilated into hypotheses about intellectual competence.

In part, learning to be a problem solver means acquiring a problem solving personality. One of the first characteristics of a good problem solver is that he interprets an unusual number of daily situations as problems - that is, as situations where his problem solving skills may be relevant. One of the first steps a good problem solver takes is to identify a given problem as "do-able" or "undo-able", and next as "easy" or "difficult". Then appropriate solution strategies are selected to fit the initial appraisal. However, it is quite obvious that people who are good problem solvers in one context, in one type of situation, or in one discipline, may be average or below average problem solvers in another.

Research has shown that children may well form severe personae for different situations - family, peer groups, school. The child's "personality" is different at different times, in different situations, in different mental and physiological states, as a function of cognitive, social, and emotional load, and as a function of his particular agenda of the moment. Cognitive performance is moved upward and downward by load factors in the child and in the situations - e.g. noise, emotionality, distraction, confusion, shyness, anxiety. So children fluctuate in their apparent ability depending upon time and place. Psychological descriptions of people have for a

long time centered around the premise that people have traits, personality traits and cognitive traits, that endure and that are manifest in all situations. This notation is currently under strong attack, particularly in personality theory.

The above sorts of factors suggest another set of benefits associated with group problem solving activities. For example, Triandis (1976) lists the following points:

a) Imitation: There is a good deal of learning that takes place when humans see connections between what other people do and rewards they receive. (One of the characteristics related to this factor is the viewpoint that behavior is not viewed as being reliably followed by particular outcomes good or bad.) Triandis calls this "ecosystem distrust". Seligman (1975) calls it "learned helplessness". Lescart (1976) refers to the factor as "locus of control".

b) Groups create satisfactions with certain activities. An activity which has low intrinsic value, when the learner is alone, may acquire a much higher value when the learner is in a group. Group norms, or roles adopted by the learner in a given social setting, or interpersonal agreements between the learner and important others, can change the desirability of activities.

c) Self-esteem is important. The learner's view of the self, particularly beliefs that certain kinds of learning are possible and desirable, is important in learning. Among learners who think that they are not capable of learning some material, or who do not see

the desirability of learning it, this factor severely reduces learning. A person's self-concept is in part formed through interaction with other people.

11. Focus on "Real" Probelms:

What makes a problem real as opposed to artificial? In this paper, space limitations do not allow a thorough answer to this question. Some attributes which should be considered are apparent from the factors that were mentioned in the preceding sections.

Real problems do not have to be job oriented, and they probably do not even have to involve real world objects - at least not if the real world is defined from an adult perspective. A fanciful or imaginary problem having to do with "Star Wars" may be meaningful and interesting to youngsters. Or problems using poker chips, cuisenaire rods, geoboards or other materials can be quite appropriate for young students. But the problems should not ignore all of the types of modeling processes, information retrieval processes, resource utilization process, etc., that were mentioned in the preceding sections.

As scientists we know that real problems seldom fall neatly into diciplinary categories. A problem which begins as a measurement problem may turn into a probability or statistics problem which may in turn become a social issue or a legal problem ... The same is true at elementary levels. For example, when a sixth grader uses arithmetic to solve problems in everyday situations, problems are seldom simply addition or multiplication or division problems. They often involve combinations or

sequences of basic arithmetic operations. In real problem solving one of the first and most important steps is to identify the appropriate computational structures to deal with a given situation.

In real problem solving, the exact nature of the answer (if one exists) may not be clear. Many levels of correctness may be possible, and many paths to solution may be appropriate. Or, in many situations, the goal may be to understand a problem, not to solve it. Or, in other situations, what is involved is more like a project than a problem. That is, a sequence of problems may arise, with each temporary solution leading to new, more sophisticated problems.

Time constraints may or may not be a factor. Often, good problem solvers, just like good runners, know how to pace themselves. They know when to relax and when to exhaust themselves. Many other factors could be mentioned. I hope some of these will be topics of discussion at the Osnabrück Conference. The above factors and others are discussed in greater detail in the upcoming monograph on Applied Problem Solving.

Summary

Originally, I became interested in space and geometry research because I was interested in finding ways to concretize mathematical concepts - any mathematical concept, not just spatial or geometric ideas. I was interested in Piaget-inspired research analyzing the development of mathematical ideas, but I was convinced that (especially for formal operational youngsters)

Piaget's focus on the "structure" of concepts did not give an adequate description of the "dimensions of meaningfulness" for most mathematical concepts. The operational or relational structure of a concept did indeed seem to be important characteristics determining the difficulty of the concept but figurative characteristics also seemed important. Furthermore, certain "applied problem solving processes", such as the ones described in this paper, were clearly important - both when children tried to use mathematical ideas in real situations, and when concepts were in the early stages of development. Geometry seemed to be an ideal context to study modeling processes and the influence of figurative characteristics of ideas.

I was also interested in "mathematics laboratory" methods of instruction. I was impressed by the fact that so many laboratory theorists claimed that mathematical concepts developed through interactions with the environment - and that these interactions were of two basic types: (a) interactions with concrete materials (or lower order "concrete" ideas) and (b) interactions with other people.

Some theorists, like Dienes, have given reasonable explanations of what students are supposed to get out of interactions with concrete materials. But, no one seems to offer a comparably convincing explanation of what students are supposed to get out of interactions with other people. What kind of interactions are supposed to be beneficial in laboratory situations? How are these personal interactions related to the interactions with materials?

I am pleased to say that some of my recent work in group problem solving settings has helped clarify my thinking on several of these issues. However, it is premature to report the results of this work. For the purposes of stimulating discussion at this conference, it seemed preferable to describe some of the factors I believe are important in applied problem solving.

I would like to close this article with a few brief comments about a theoretical model I have been using - and one which I believe is particularly relevant to the kind of research variables described in earlier sections of this paper.

One of my colleagues, Don Saari, an applied mathematician, has developed a mathematical model for the theoretical perspective I am taking (Saari, 1978). The model was originally based on some relatively new ideas in a branch of mathematics called catastrophy theory. Specifics concerning the mathematical model are probably not relevant to mention here. It is sufficient to mention that catastrophe models are useful:

a) When several continuously changing variables give rise to abrupt changes (e.g. insights, etc.) in the behavior of an organism or system,

b) when behavior at any given moment depends not only on the strengths of the control variables but also on the previous relative strength of the control variables,

Saari's model has the additional characteristic that

c) problem solving ability (or concept acquisition ability) at any given moment depends not only on available cognitive structures but also on certain social/affective structures.

The above three conditions are typically present in problem solving situations.

The model is especially suited to describe the evolution of adaptive systems - or structures. Therefore, because the characteristic feature of mathematics is its "structure", the model is particularly relevant to the development of mathematical concepts. To put it very simply: learning a system of things is usually quite different from learning each of the things in isolation. That is, the whole is usually more than the sum of its parts. For example, Schoenfeld confronts precisely this issue when he points out that mastering a system of heuristics in isolation. My research suggests that this idea applies equally well to the acquisition of any mathematical system. Mathematical judgements or mathematical ideas almost always require a student to master a coordinated system of relations, operations, or transformations. This is why "structured learning theorists" as diverse as Piaget (who focuses on "written-concept" structures), and Ausubel, Bruner and Gagne (all of whom focus on different aspects of "between-concepts" structures) have all been particularly relevant to mathematics learning.

Saari's model is useful because it gives a precise mathematical description of previously ambiguous terms like assimilation, accomodation, intuition, etc. In

this way it draws together and coordinates useful "structured learning variables" from theories like Piaget's and Ausubel's - as well as a number of useful "process" variables from computer simulation/information processing theories like Greeno's or Kr·tetskii's. The model also defined "problems" and "problem solving processes" in such a way that they are closely related to concept formation, to modeling processes, to information retrieval processes, and to other factors that were mentioned in this paper.

REFERENCES

Carpenter, T.P. Analysis & Synthesis of Existing Research on Measurement. In R. Lesh (Ed.) Number & Measurement: Papers from a Research Workshop. ERIC/SMEAC, Columbus, Ohio, 1976.

Case, R. Learning and Development. Human Development, 15, 1972.

Greeno, J.G. Cognitive Objectives of Instruction: Knowledge for Solving Problems and Answering Questions. In: Klahr, D. (Eds.) Cognition and Instruction, Lawrence Erlbaum Associates, Hillsdale, New Jersey, 1976 b.

Greeno, J.G. Indefinite Goals in Well Structured Problems. Psychological Review, 1976 a.

Hayes, R. & Simon, H.A. Understanding Written Problem Instruction. In: Gregg, L.W. (Ed.) Knowledge of Cognition, Lawrence Erlbaum, 1974.

Kieren, T.E. The Rational Number Construct - Its Elements and Mechanisms. Unpublished manuscript. University of Alberta, 1978.

Kohlberg, L. Stage and Sequence: The Cognitive-development Approach to Socialization. In: Handbook of Socialization Theory and Research. Goslin, D. (Ed.) New York, Rand McNally, 1969.

Kosslyn, S.M. & Schwartz, S.P. A Simulation of Visual Imagery. Cognitive Science, 1977.

Krutetskii, V.A. The Psychology of Mathematical Abilities in School Children. University of Chicago Press, Chicago & London, 1976.

Kuipers, B.J. Modeling Spatial Knowledge. Cognitive Science. 1978.

Landa, L. Algorithmization in Hearing and Instruction. Engelwood Cliffs, N.J. Educational Technology Publications, 1976 a.

Landa, L. Instructional Regulation & Control. Engelwood Cliffs, N.J. Educational Technology Publications, 1976 b.

Lescourt, Herbert M. Locus of Control. Current Trends in Theory and Research. Halstead Press, Lawrence Erlbaum Associates, Hillsdale, New Jersey, 1976.

Lesh, R.A., Kantowski, M.G. & Mierkiewicz, D.B. (Eds.) Applied Mathematical Problem Solving. Columbus, Ohio: ERIC/SMEAC, 1979.

Mead, G.M. Mind Self and Society. University of Chicago Press, 1934.

Newell, A. & Simon, H.A. Human Problem Solving. Prentice Hall, 1972.

Pascual-Leone, J. Metasubjective Problems of Constructive Cognition. Canadian Psychological Review, 1976.

Saari, D.G. Cognitive Development and the Dynamics of Adaptation: Accomodation. Unpublished manuscript Northwestern University, 1978.

Seligman, M. Helplessness: On Depression, Development, and Death. San Francisco, California, Freenon, 1975.

Triondeo, H.C. Social Factors in Science Education. University of Illinois, Urbana-Champaign, unpublished manuscript, 1978.

A PORTRAYAL OF TRADITIONAL TEACHERS OF MATHEMATICS IN AMERICAN SCHOOLS

Jack Easley

This paper includes selected findings from the CSSE (Case Studies in Science Education) project at the University of Illinois, which relate to mathematics teaching. The National Science Foundation has supported the development and dissemination of many curriculum projects in natural sciences, mathematics and social sciences over the past two decades, but found themselves much criticized for exerting too much federal influence on local schools. A few years ago, it undertook a group of studies of educational needs. CSSE was one of these studies. It was contracted to the University of Illinois to study in-depth the teaching of natural sciences, mathematics, and social sciences in representative schools, covering grades K-12. It was conducted jointly by CIRCE (Center for Instructional Research and Curriculum Evaluation) and the Committee on Culture and Cognition. Bob Stake and I were co-directors of the project. Here's a sketch of that highly interdisciplinary project from our final report.

> Case Studies in Science Education is a collection of field observations of science teaching and learning in American public schools during the school year 1976-77. The study was undertaken to provide the National Science Foundation with a portrayal of current conditions in K-12 science classroom to help make the foundation's programs of support for science education consistent with national needs. It was organized by a team of educational researchers at the University of Illinois.
>
> Eleven high schools and their feeder schools were selected to provide a diverse and balanced group of sites: rural and urban; east, west, north and south; racially diverse; economically well-off and impoverished; constructing schools and closing schools; innovative and traditional. They were finally selected so that a researcher with ample relevant field experience could be placed at each. To confirm findings of the ethographic case studies and to add special information, a national stratified-random-sample of about 4,000

> teachers, principals, curriculum supervisors, superintendents, parents, and senior class students were surveyed. Survey questions were based on observarions at the eleven case study sites.
>
> The field researchers were instructed to find out what was happening, what was felt important, in science (including mathematics and social science) programs. On site from 4 to 15 weeks they were not required to coordinate their work with observers at other sites. Questions originally indicated by the NSF or identified early in the field were "networked" by the Illinois team. Efforts to triangulate findings were assisted by reports of site visit teams.
>
> Each observer prepared a case study report which was preserved intact as part of the final collection, and later augmented with cross-site conclusions by the Illinois team. The case took 18 months and about 6 research-person years to complete. (CSSE, p. 19:ii)

This paper reviews evidence relevant to the apparent communication failure between many teachers and specialists in various aspects of mathematics education. After quoting a preliminary statement of this problem, I'll then report what we found that relates to various facets of and social mechnisms underlying this communication blockage, using mainly quotations from the pages of our final report. (Just as people use the Bible, there are many possible interpretations.) After quoting briefly from two other studies supporting my general view, I'll then turn to a brief consideration of the possibility of reducing this communication problem and consequently increasing the possibilities for improving mathematics learning through the collaboration of communicating specialists and teachers.

This project represented a change for me. Unlike the studies I had done earlier on children's thinking in science and mathematics, this one focused on teacher's thinking about teaching and on social systems in which teaching was done. In a sense, we're all teacher educators in that, at least, our research might someday be

disseminated to teachers. While we might agree readily with the view that a teacher needs most to know his pupils' ways of thinking, it is not so popular to believe that a teacher educator needs most to know the ways of thinking of teachers or future teachers. Here's what we said in our final report:

> It was our conviction before we started this study, and we became even more persuaded during the doing, that research on the context of instruction rather than research on the learner is more likely to yield insights into the ways of improving the quality of education that is offered. What research on the learner tells us is the vast number of ways people differ, and how greater experience increases those differences. What research on the pedagogical processes, the administrative processes, and the social-political background tells us are the obstacles to learning and the obstacles to changing the opportunity for learning. The cognitive-personal obstacles to learning are formidable, and also not much amenable to school control. The social-political obstacles are also formidable, and also not much amenable to school control, but somewhat more. (CSSE, p. 19:26)

In an effort to focus our attention before we began to collect data from our eleven sites, I had written:

> The ways in which curriculum issues, basic human concerns, and social mechanisms in schools are related are apparently complex. If we are to avoid entanglement in the mesh of these relationships, we need to understand the mechanisms that connect basic concerns on the two sides of a widening gap between universities and schools with the issues that divide educators in both kinds of institutions. Let us try to chart the network of concerns and to lay the groundwork for a more precise study of these mechanisms. We begin with the principal issue which forebodes poorer relationships between the schools and universities.
>
> It is widely known that a large and vocal percentage of inservice teachers assert that too many of their courses at the university (especially education courses) were found to be of little worth once they were on the job. The university courses are called "idealistic", "theoretical", and said to "rarely, if ever, contribute to work in the classroom". The "real world of the school" is said to be filled with so many children "who don't want to learn", and who "are incapable of grasping the subtle ideas and relationships of theory", that old fashioned or other methods

disapproved of in colleges of education are really necessary. What works in schools, we're told, is to give students specific assignments and let them know the consequences of not doing them. Later, they will appreciate what they have learned in the process of carrying them out but did not appreciate at the time.

There may be just enough truth in this set of beliefs to make them very credible to a harried teacher. But I also know from my experience as a student of children's thinking that, through careful listening, observation, and active probing into a child's mind and background, I can understand how his or her very negative responses to school are conceptually intelligible, functional, and at times so advanced intellectually as to put the school curriculum makers to shame. For many disinterested students and so called "slow learners", school tasks are demonstrably not worth doing. In short, there is from my point of view an impasse which arises from the way pupils and teachers understand themselves and their roles. I also know this impasse from personal experience as a teacher of classes from kindergarten through graduate schools. Although I have never had to teach either the same children all day, five days a week, or worse, five different classes a day, five days a week, I know there are classes I can't teach - not because I don't know the subject matter, but because I don't understand the students. I am simply amazed that some public school teachers exist who can teach such classes all day every day and who conscientiously prepare themselves daily for six or seven hours of continuous confrontation. Some of these teachers I've met are single (nuns, bachelors, unmarried women), others are married men whose wives take care of the house and children, others are hyperenergetic people (organizers of teachers' locals and student field trips, bubbling with contagious enthusiasm for life). But I can't see how anyone can do what educational theorists would consider a good job of teaching in one of our public schools, just as a routine job, 8:00 to 3:30 plus an hour of preparation, 40 hours a week. Many teachers live this confrontation only as a daily routine. With luck, a few of their students are self-motivated, read a lot, catch on easily, and become scholars.

Now this point of view on schools and teaching not only explains why there is so much time and energy devoted to "administrative trivia" and so little to discussion of ideas, but it also explains why there is a widening gulf that separates the concerns of learning (universities, research laboratories, libraries, art colonies, etc.) from schools. Instead of being a bridge to the "world of ideas", teachers find themselves managers, wardens, and monitors of children - most of whom don't care to know what goes on the centers of learning. But that doesn't mean these children aren't interested in ideas. Where do children learn about the excitement of modern astronomy, of nature, of other scientific frontiers. From the magazines and television, especially Startrek,

> The Wild Kingdom, and sometimes Nova. Reading Scientific American keeps me in a continuous state of excitement about what is going on in university science, but schools do little to prepare students to be that literate. The students notice that they are only offered the prequisites to exciting science (drawing onion cells, learning metric units, computing with decimal fractions, preparing oxygen or carbon dioxide, reading a chapter in a psychology book about Piaget, studying the law of supply and demand, etc.). These are usually dull and boring and lack the excitement of TV science and science fiction. Science enthusiasts can buy kits (if affluent) to make interesting devices and learn technical information far beyond the curriculum. So they, too, are fored with school science. They may prefer vocational courses where they learn skills they can use in their hobbis. But mastering electrical gadgets, even mechanical gadgets used to be science. (cf Leonardo and Faraday)
>
> Of course, teachers are not well prepared by universities with techniques to make ideas exciting because so many university professors are operating under the assumption that university students are already motivated and want to learn. That may even be true of many, but operating this way still doesn't prepare these students to make ideas exciting to children who find school boring, or to those who have accepted the kind of schooling that is just a matter of getting your work done with no discussion of ideas. (Easley, 1976)

While all research must choose its level of complexity, the problem is acute in case studies. We wrote:

> People sometimes ask for a clarification of school goals, and that is no more and no less than they are interested in. It can be as simple as that. And people sometimes ask for a clarification of school goals, and they are there pointing out that children are unwilling to learn, that parents are unconcerned about what the children do, that the learners are unprepared for learning, that teaching is over-demanding, etc., etc. And it is as complex and diffuse as that, or more. And it may be the same situation and the same people and the same moment on the clock. And it is not that those who see the complexity, "see through" the simplicity; because the situation is not complex until the reality of simplicity is no longer sufficient. It often is. (CSSE pp. A: 5 - 6)

We have asked large numbers of questions so that individuals could tell us how other individuals especially large groups of individuals act or feel. They found this quite difficult to do. They said they did not know. This does not mean that there was no sharing in a deep cultural sense but that it was not an explicit, conscious sharing.

1. Teachers have their own values to teach.

One aspect of the saparation between specialists and traditional teachers is portrayed in one of the CSSE site visitors' reports. Mathematician Hassler Whitney wrote:

> Loraine had written $\frac{3}{5} = \frac{2}{20}$, and asked me if that was right. I drew a rectangle, divided it into fifths, and asked if she could show me three fifths of the rectangle. She shaded this. I then asked if she could cut the rectangle into twentieths. With some trouble, she cut each piece in half, then in half again. Now she found that she could count the twentieths, and got twelve. She expressed new understanding. (Of course, I was using a remedial technique that is supposed to give understanding, and does something in this direction, but by itself is not lasting - it was too much from me, not from her; I wanted to see if she could get the sense of what meaningfulness could be. She is going into nursing school; understanding is much more important than algorithmic knowledge here.) (Emphasis added by CSSE staff.)
>
> I talked to her a couple of minutes to see if she could get a sense of the need for and meaning of security, so she would not need to have to assent to each thing.
>
> This is one of the greatest problems of present day "back to basics", as I see it. The attention goes so thoroughly onto doing the right thing, hence learning and remembering, that there is not at all sufficient time or energy to grasp what is happening in your own terms. Then with teacher or helper not around, students feel quite insecure or lost. In particular, they do even more poorly on basic skills than in the more traditional classes. This is certainly true very largely, though some are better than others at memorizing and following rules. Even these are most commonly unable to solve story problems, except more or less by rote, if in a familiar form.

> Discussion with Mr. Blue: He has already spoken of basics as the greatest need. I spoke of my experiences with Ramondo, and how understanding might help. His reaction was to get turned off; he had probably had too much of this kind of talk, and long experience in failing to see better results from "discovery". "Drill it into them" was his comment; he had learned that this was the only effective approach. (CSSE, pp. 16: 3 - 4)

Another aspect of the general problem, which affects the technologists' approach to mathematics education, was described by another site visitor, retired curriculum lecturer at Leeds University, Frances Stevens. How teachers use technological aids (whether textbooks or computers) is determinded by their own personal values. She wrote:

> Students taught - and, in these two schools, for the most part very efficiently - within a conventional and conservative framework are following two curricula. The first is the body of established knowledge and procedures to which the learning encounter is intended to be instrumental. The second is the personality, which of course includes individual preferences and value-systems, of the teacher, communicated in the learning encounter. Students of the teacher just mentioned must have learned, besides the facts and processes contained in the syllabus, his passionate concern for the careful discriminations and unremitting logic as well as his commitment to an austere discipline that shuns delights and lives laborious days. (CSSE, p. 16:9)

This seemed to be at the heart of teachers' struggle, a struggle with which they get little outside help. Worse still, the few minutes they could devote to this struggle were increasingly conscripted for the formalization of instruction, such as objectives-writing and testing. It reflected and grew out of struggles with their own studies or at least that part of their studies in which they could take personal pride. It involved an intimacy, a sharing of pride and value-commitment, with their students, within the context of instruction in math, science, or social studies. (CSSE, pp. 16:9 - 10)

As teachers describe their difficulties in teaching, e.g., the loss of traditional classroom control mechanisms, their personal values become clearer. Our summary chapter reports:

> Especially in communities such as ARCHIPOLIS (see p. 9:25) and WESTERN CITY (see p. 7:27) where many children needed extra teacher attention, where school kindled little spark and where authorities struck little fear in youngsters, teachers were suffering the loss of two traditional control mechanisms: grouping students according to talent and motivation, and isolating the majority from the slow, the difficult, and destructive. It was not that teachers could no longer "track" students - in fact it was apparent they could do so, in ALTE subtly, in RIVER ACRES, and WESTERN CITY openly - but they all had "so many problem children" and little way to help them or even to keep them from having an adverse impact upon classroom activities. (CSSE, p. 19:9)

Teachers have to keep trying - that's one value they hold - and stay on top. I reported this conversation with the teacher after observing a junior high school class:

> She remarked right away, apparently aware that the stuck pupil at the board was a scene that needed explaining, that there were several students who had not developed enough personal resolve to tackle their work when the going got rough. "I don't know what's going to happen to them," she said, "but I can't give up trying to get them trying harder." "What if they don't understand?" I asked. She replied, "I can't accept that as an answer. They have to learn to keep trying." (CSSE, p. 16:15)

With decimal fractions, a scenario used in one of our questionnaires presented the case of Tom, whose problems resembled those Erlwanger described in the case of Benny (1973). One question asked, "If you were Tom's teacher, how would you deal with this problem of his?" A junior high school math teacher responded as follows:

> I would not have gone to the rectangle method. I would have discussed place value instead. Also, I emphasize that rules like laws must be followed if society is to progress and communicate.

Another wrote a lengthy reply which included these remarks, making a religious metaphor of the confession of error.

> *I find out on the high school level, a student with such a background is impossible to teach until he can admit that maybe he may have been using erroneous logic. That's very difficult to elicit from the student. In our school, I teach all low-proficiency students.... Once I can get them not to be ashamed of deficiencies in addition facts, for example, progress can be made doing something about these deficiencies. Sometimes this takes a year for some students, if this conversion takes place in a group, the results are even more fantastic. Then we are in this learning process together with group support.* (CSSE, p. 16:37)

When all responses were sorted out, and we compared K-6 math supervisors with junior high math teachers, we could understand the skills emphasis of teachers more precisely. For example, noticeably more teachers than supervisors recommended training in "abstract" symbol manipulation rules, the idea of equivalence, and teacher activities of a traditional sort (drilling, explaining, questioning). But more supervisors than teachers recommended pupil activities of manipulation of concrete objects, discussing, calculating mentally, or arrangements other than the usual teacher-dominated classrooms scene. Those data from 81 teachers and 116 supervisors reflect, I think, the norms of the professional groups to which they belong.

2. Mathematics is not value-free.

Mathematics, as subject matter, is not, however, somthing that can be dissociated from control and value questions.

Computation ability, like good diction and polite manners, is a social badge of membership in society.

> With the national emphasis on "the basics" and on vocational preparation, mathematics was getting increased attention. The result was an almost exclusive concentration on computation, from second grade math to that in senior year. Many schools, such as those in URBANVILLE and WESTERN CITY had a computation test which had to be passed sometime prior to high school graduation. There was little feeling for the importance of mathematical concepts, as a legacy of the "new math" efforts. The attitude among many math teachers was that new math was too difficult for youngsters to learn; it allowed them to drop behind in computation skills. The commercial world increasingly required less in the way of computation (e.g., by providing cash register keys with pictures of sandwiches, automated inventory card, hand calculators), but the belief in the need for computational skills was strong. According to the prevailing social and educational ethic, a disproportionate time should be spent on computation. Much of the remedial teaching in mathematics in the URBANVILLE school and elsewhere was being done by non-mathematics teachers reassigned for various reasons. Some teachers said they needed materials more suited to older students, those slow and little motivated. (CSSE, p. 19:3)

Thus mathematics, at the hands of the traditional teacher, is more a matter of skills than of ideas. Drill may be linked with the orderly society and advancement of our country. One teacher told us,

> *I still think America came farther and faster than any nation in history under the old method of teaching, where we had some discipline in the classroom, we did some drilling because it was what teachers deemed was necessary, we didn't have to try to justify all that we did.* (CSSE, pp. 12:33 - 34)

Initiative and responsibility, are primary concerns te teachers have about pupils. One junior high school science teacher from England said to me, with reference to some test items we were discussing, "Many 14 - 15 year-olds won't try anything to test whether the iodine goes both ways through the cellophane. They just sit and stare at it. I know this item measures self-reliance.

If you think these other items measure the same thing then I'll know what they measure."

This point involved a shift in his vocabulary from talking about the process of science to talking about pupils' behavior in the typical way traditional teachers psycologize or moralize about them. Teachers know both academic and moralizing talk, use both with reference to subject matter topics, but their problems tend to be conceptualized in moralizing terms, and they find few if any specialists of any persuasion willing to talk with them about or help them with ways of socializing students through subject matter.

Summarizing this distinction we wrote:

> But one thing clearly was happening. The curriculum was becoming more oriented to general skills (adding, reading, observing, looking up information) and less oriented to subject matter content. This was partly because a skill item appeared to be more basic and universal than a content item and therefore "relevant" to a larger stretch of the instruction and more likely to be good preparation for later courses. Perhaps research and development in the science content areas might help to return some of the attention of both instruction and testing to the subject matter of sciences. (see CSSE Chapter 15)
>
> It is noteworthy that we have not been able to provide standardized testing instruments which are focused enough to note when a teacher has spent an extra two weeks on a topic, but not so narrow as the present lesson-specific criterion referenced tests. There are many problems with testing, and developing more tests may add to the burden of instruction and the risk of further imbalance to educational opportunity, but the present negligence of testing for subject matter sophistication seems to call for research attention.
>
> Research on Use of Science Instruction for Socialization. The uses teachers made of science subject-matter and instructional materials for socialization, that is, the inculcation of values as described in Chapter 16, were subtle and pervasive and often perplexing. It appeared to some to be a means-ends reversal, at least to those of the rational world of science, where scientific

> inquiry can stand as an end in itself. However, from the point of view of sociology and anthropology, educational institutions function primarily in the transmission and preservation of societal values. Thus the context of the general educational programs, including ritualistic activities, served primarily to establish the attitudes and habits of behavior in youth which become the admission passes to membership in adult society.
>
> In a technological age, when vocational training becomes more and more specialized, there is a possibility that general education in scientific knowledge may function more and more as a behavioral badge of eligibility for employment, further educational opportunity, and various privileges of the society. The strong suggestion in Chapter 16 that teachers recognized this function and wanted help in adapting newer instructional materials to these ends needs further educational innovations, as they appeared in this study to do, they need to be more throughly understood than they are now. (CSSE p. 19:27)

This distinction explains the common attitude toward the use of miniature calculators in schools in America.

> When the subject of hand calculators was raised with the group of URBANVILLE parents, the level of emotional response suggested that the topic was not one of utility but of ritual, which typically has a higher affective component. They said:
>
> > *I think using hand calculators is an awful shame because that's why our brains are going so lazy.*
> >
> > *Calculators should be forbidden in math classes, at any level, because even in higher algebra and trig processes, the learning process is by doing it.*
> >
> > *Kids are lazy; people are lazy. I'm lazy - and you're going to get by with as little as you can.... I think they should be required to take a little more. The program is getting watered down.* (CSSE, p. 12:33)

Science and mathematics, while often thought of as value free, are seen as heavily laden with social values in the eyes of the traditional teacher. That lists of curriculum goals poorly reprsent teacher goals was shown again by Helen Simons, site visitor from the University of London, Institute of Education. Summarizing a fifth-grade lesson

she had observed based on the Elementary Science Study unit, "Peas and Particles". Helen wrote:

> What I find interesting about this teacher's style is her mixture of openness of acceptance of where the children are at, intellectually, and certainly about what she wants them to know. She left them to work out their own strategies first; listened to these; made sure that other children understood the method each group had used by interjecting questions like, "How did you get that?"; "Wait a minute. I don't understand"; "I don't know what that means." Then she evoked alternatives. If the children did not raise them (mostly they did) she would guide them to alternative ways of resolving the problem which she would then explain as she did with sampling which John had not used. She is always in control and stays with the question until she is satisfied that they have understood it. Her tone was consistent throughout. If children said nothing, that seemed to be O.K. If they appeared to be on the "wrong track" she would always say "O.K., all right,", and try a different strategy to push them towards another method or the one she wanted them to learn. If someone gave the wrong answer in the same tone she would simply say, "no" and look for the right answer from someone else. There was no hint of criticism in her voice. Occasionally she did check up on those who appeared not to be paying attention by saying, "What did she say? I did not quite hear"; or, "Are you guys paying attention over there?" But I sensed this was more an attempt to get them to focus on the task and learn than to censure their inattention.
>
> What is obvious to an outsider is her commitment to the brightest. In discussion, the articulate and confident are heard first. There was John, who was asked to offer his explanation. When he had finished, she responded to one of the brightest girls ... and was in the process of listening to an explanation of that girl's second problem (despite an objection from one of the other groups that not to hear them first was unfair) when the bell rang. She did not pay much attention to those who did not have the immediate reply. One senses that she gets a lot of intellectual excitement and rewarded from hearing the brightest talk through their discoveries or guiding them to understandings she thinks important. (CSSE, p. 16:17 - 18)

Still another interesting point came out when Helen Simons asked this teacher if she saw a difference between social science and social studies. She replied:

I don't think I differentiate here. I think they overlap. I think ... I guess social science would be concerned with teaching values ... some important concepts about man and his environment whereas social studies is technically geography - rivers and their tributaries ... and possible geographical concepts like that I think I would classify that as social studies. But social science you really have to learn about living in a community and having values. That to me is a science. (CSSE, p. 16:18)

And the lesson just observed? She hesitated and then suggested that when you come into a new area and have to decide what apartment to live in you estimate and decide on the basis of this or that observation of number of families, trees, children, etc. (Peas and Particles is about estimating numbers of things in large collections.) . . . what you prefer . . .

So I think there is a lot of social science there . . . being able to estimate and apply this and transfer this learning to something like that . . . (CSSE, p. 16:19)

3. Teachers are not prepared to understand learning.

Even with the understanding that comes to teachers with experience, much that happened each day in classrooms had an air of mystery about it. Who knows how Harold managed to learn that skill? It is impossible to believe that Linda can no longer do "Sevens", but she cannot, and she says she never had them. A RIVER ACRES teacher put it this way

When the light goes on for a kid I ask, "What did I just say? Tell me, so I can use it next period." It doesn't work next period of course. Today we were doing the expansion of binomials. We had done the usual background work on this. So I was giving them Pascal's triangle and some of them caught that rhythm just as soon as I put it down. Others were saying what, hey, where are those numbers coming from? So, we work on it together and alone. We lost some of them and came back and did some more - then one said, "Oh, I get it!" Then a couple others. I don't know really how that happens. (CSSE, p. 1:106)

Traditional teachers are also very ill-prepared for pupil forgetting. Observer, Terry Denny, was told:

What I want you to know is that ALL the kids except block 1, and some of them too, have this mysterious "forgetting disease". Last week I had the "2's" borrowing and carrying and I was so happy and we were all dancing. After Thanks-giving they came back and acted like they had never heard the word "borrow". Borrow!? Teachers live with this. They see it every day and get used to it, so we don't talk about it. But it's there teaching math; day to day. (CSSE, p. 15:9)

They are also ill-prepared for the social heterogeneity of their classes. Jacqui Hill reported:

In the junior high schools the salient issue was "mainstreaming": not so much random mixing of abilities, but the mainstreaming of youngsters with behavioral problems, "juvenile delinquents" as they were sometimes referred to. This was no small concern. Children were being returned to the schools, and to the same classrooms, by the court. A federal court assured these children the right to reenter those classrooms. Following this legal mandate of the courts, the teachers of one of the junior high schools submitted a petition to the teachers' union to initiate a "class action suit" on their behalf. The stress ran deep. One day I entered the teachers' lounge with Ms. Odom during her free period. I saw an older man with close-cropped hair sitting at the end of the table nervously smoking a cigarette. He held his head with one hand, stroking back across his hair from time to time as if to relieve it of pain. When introduced to me as Mr. Thomas, math teacher, he asked if I were there to introduce a new curriculum. I said "no, not this time". I was there to find out what was happening now in science teaching and what teachers thought of it, good and bad. He said:

You get kids and they don't know what they should know to do the work. Since they can't do the work, they act up. They don't want kids to call 'em dumb, so they act up to cover up the fact they can't do the work.... They can't subtract and multiply. They know theory and sets but can't subtract.... New math seems to have done that; the paper says high school grads can't even read!

Q: Do you want to go all the way back?

Mr. T.: No, not all the way, but some.... Oh, I got a headache just looking at this school this morning.... A kid said to me ... pardon me ... he said to me, "Kiss my ass!" Teachers have no rights anymore, only kids. (He rubbed his aching head and *drew deep drags from his cigarette.) Kids are all mixed up now.*

Q: Is it the size of classes?

Mr. T.: No, size isn't it. You can have fifty who want to learn and still have a good class. We're not allowed to group kids homogeneously. You'll have kids in a grade who can do the word, but some who are two or three grades behind in reading; you have to individualize or group. You can't teach otherwise. But you're just a security guard. (Then) ... Come to my second period class, I'll show you what good kids can do. (CSSE, p. 19:4)

4. They cry for HELP!

Terry Denny commented on one kind of help teachers asked for:

> Teachers did not call for research or for evaluation of instructional materials or the curriculum. They want help, now. They have three widespread concerns. My notes contain thirty-seven separate pleas (not all elementary) for materials, procedures, aides, or supervision for slow children in mathematics. The harshest self-criticism made was in the mathematics instruction of students in the lower levels. (CSSE, p. 1:32)

Teachers there called for inservice, cross-grade meetings and vertical communication to replace "failure" of supervision to coordinate the curriculum (CSSE, p. 1:38 f).

The "best" teachers have problems doing what they want to do.

The enthusiasm teachers had for positive responses from "weak students" was such an important part of their concern for socializing students that another example (besides the case of the student whose English was poor) is worth considering. Frances Stevens quoted a history teacher as follows:

> "They're so much geared to giving answers," said the teacher (alluding to the answer-question approach in the class I had observed) "that I thought I'd give it to them the other way round." How much they were indeed "geared to giving answers", was shown in another part of the same lesson. In answer to the question, "What were two of America's big problems in the post-World War II years?", one boy said, "The world didn't have enough love." This response provoked a gale of laughter from the class and slightly nonplussed the teacher who commented mildly that this was connected with a lot of problems. When I later referred to this interchange, the teacher's face lit up. "Oh, that was one of those beautiful moments," he said. "It's like music - you just know when it's right. He's not very bright and d'd you notice how David (a bright student) put him down? But I think they know how I feel about these things - I guess that's the most important thing." Was it? I deliverately recalled the incident again, when interviewing one of the students. The boy laughed. "Oh, that!" he said. "He just said that because he didn't take it from where he was supposed to. He made that up." (CSSE, p. 16:11)

Clearly, students were making a rigid system of the standard way the school rewards them. How impotent is this teacher's desire to support certain values against that system!

5. Administrators are able to be of little help.

Terry Denny wrote,

> One of the storms that recurs regularly for the administration is the practice of grouping. How many levels; what criteria to use; and what are its effects? The conventional representations are made. There are calls for more instructional levels, for as many as seven in each grade in each subject. There is a top administrator who wants fewer instructional levels, "two would be about right".
>
> Others feel the administration is caught in a responsive rather than a leadership mode. They assert that the pace of change in RIVER ACRES is more than it can handle. "Who could handle it?" I asked. "A young, sensitive Texan who could make us proud of what we are - and who ate hurricans for breakfast." (CSSE, p. 19:33)

6. Change comes slowly.

There are many other studies confirming the general picture we have developed of teachers, but few that focus especially on mathematics teachers. Mathematician-psychologist, Klaus Witz, of the Committee on Culture and Cognition at Illinois, has written a paper critically examining social science perspectives on social stability and change. He takes as one example, a study of students entering training as high school math teachers by George Shirk of Toledo University, Ohio. Witz writes:

> Reading these case studies one cannot fail to be struck by the fact that in each person, innumerable behavior patterns at many different levels have subtle affinities for one another and coexist as if in some strange harmony. For example, one of the subjects, Tom, judging by term papers of his on 'what is mathematics' and on 'the essence of teaching', conceives of mathematics as an ordered system of propositions and considers the basic role of the teacher to be to establish in his students belief in these propositions. This has affinities with his tendency to interpret a student's failure with a particular point as indicating a need for more explanations by the teacher on appropriate principles that had been covered earlier. It also has affinities with his tendency to keep on going over the previous principles persistently, and trying not to get sidetracked, until the student performs as he should. (For details, see the discussion of some specific classroom episodes on p. 45 ff. The term 'has affinity with' here is essential; alternatives such as 'is supposed by' or 'implies' are misleading.) The impression one gets reading these case studies, then, is that the whole organism at the given time period is a gigantic stable organization, permeated by affinities, through and through, and thus with unbelievable cohesion. Each structure is in harmony with many others, so that on the one hand it is impossible to intervene and change a single one (it would be immediately regenerated by the remaining system), and on the other hand, even a small change propagates deeply into the whole system. (Witz, 1978, pp. 2 - 3)

Another example, permits Witz to construct a theoretical concept: divided attention.

> The point now is that during this 3 or 4 minute scene, S. is in a specific state, or condition, of consciousness: Part of her attention and consciousness is on the children, their attempts to come in and get her attention, the push of the speaker, perhaps some of what he is saying, and part is on the job she has to get done, the objects on the table, her own actions and the changes they bring about, various kinds of 2, 3 second sub- or only faintly-conscious planning ahead and trying out a possibility in her mind, etc. Behind the monitoring and the constantly ongoing sub- or semiconscious "considerations of details", there is also a continuously ongoing "wanting to get the job done", a somewhat nagging feeling of "needing to get the job done"; at the same time there is a half-conscious awareness of the subtle pressures for interaction from the kids, and a half-conscious perhaps somewhat nagging feeling that the kids should not be ignored, and that they ought to be interacted with. Together these sub- or faintly-conscious undercurrents keep her attention divided in the way described and create a specific overall condition of consciousness as she is working, which is reflected absolutely precisely, by innumerable nuances, in the character of the behavior that we see: "moves in a constrained manner to avoid interaction", "acknowledges only with small gestures, in a closed manner, without inviting further interaction", etc. I will call this overall state or condition of consciousness in S., during this particular passage, "divided attention". (Witz, 1978, p. 86)

He considers this concept in the broader context of teaching.

> We said that divided attention is, firstly, a specific, identifiable, type (condition) of consciousness, and secondly, an important ingredient of what we called the density of the classroom. We conclude with a last point. Conventional classroom management today fosters the development of divided attention in the teacher. (A teacher hired into almost any public school in the U.S. today is forced to develop "divided attention" if she is to survive; furthermore, she will develop "divided attention" because she has met it in her childhood, and is still meeting it every day in her colleagues, and in the society at large.) In this sense, divided attention is a specialization of consciousness brought about in and by our society, just like the pincer of the crab is a specialization of organic form and functioning brought about in and by nature. And just like the pincer has become fixed in the crab, so divided attention has become fixed in the teacher. Analogous specializations of consciousness, which then become fixed in the individual, can of course be found in many other professions and jobs. (Witz, 1978, p. 89)

Teacher education courses are not much help, partly because the social system and values of the school tend to perpetuate themselves through self-selection for the profession of those who place first priority on orderliness and responsibility in a chaotic world. They are likely to isolate themselves from those who think of mathematics as a creative process or who want to replace teacher control of the learning process by machine control.

Carmen is a high school pupil we met at one of our sites (CSSE, p. 16:26.4). Distraught that her junior high school brother was not learning math as well as he could because ill-prepared teachers were not assigning him any homework, she promised to teach him herself. Members of a minority family, they live in a threatening world and teachers who stand for moral principles are more attractive to her than those who promote creativity or technology. She wants to be a teacher and works hard on her studies and avoids temptations that would distract her from them. She will probably succeed in her ambition and may persist in her attitude that mathematics is something requiring drill and practice of the right procedures rather than understanding through discussion and debate. (See Lakatos, 1976.) If so, it is unlikely she will be encouraged to leave teaching for advanced study and specialist responsibilities.

7. How can communication be improved?

Traditional teachers of mathematics and specialists who promote creativity in pupils or technological models could most easily absorb some of each other's views through:

a) Fool-proof materials, not "teacher-proof" materials which carries a very negative connotation. The positive connotations of something that is fool-proof are not much discussed. Why did it take 30 years to develop the "radar range"? There can be little doubt, it had to make life easier for the harried house-person and not present a problem in itself. The Polaroid Land Camera emerged and was widely adopted by both public and scientists because it was fool-proof and solved a real problem, "Did I get a good picture?"

b) Solving problems teachers really want to solve. Teachers are tired of hearing that the best ways of teaching are the hardest. Teaching is difficult enough as it is.

c) Something important and positive must be learned by psychologists of math education about the traditional teacher. Communication cannot be a one-way street, or disrespect is increased.

I wonder if we have the resources or the motivation to do what is required to bridge the widening gap that separates us from classroom teachers and from ourselves in that function we serve when we teach what we know to our graduate students, our research assistants, our children. The traditional teacher is in all of us and comes into play in stressful situations, when we feel we have a right to demand that others go along with us only on our authority to see what benefits might follow.

REFERENCES

Easley, J.A., Jr. (1976) Issues dividing universities and schools. CSSE working paper.

Erlwanger, S.H. (1973) Benny's conception of rules and answers in IPI mathematics. Journal of Children's Mathematical Behavior. Vol. I. No. 2, pp. 7 - 26.

Lakatos, I. (1976) Proofs and Refutations: The Logic of Mathematical Discovery. Cambridge: Cambridge University Press.

Shirk, G.B. (1973) An examination of conceptual frameworks of beginning mathematics teachers. Unpublished Ph. D. dissertation, University of Illinois at Urbana-Champaign.

Stake, R.E. and Easley, J.A., Jr. et al. (1978) Case Studies in Science Education (CSSE).

Witz, K.G. (1978) Fundamental change in school, society, and general consciousness (Part I) Committee on Culture and Cognition (paper No. 2) University of Illinois at Urbana-Champaign.

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HIDDEN DIMENSIONS IN THE SO-CALLED REALITY OF A MATHEMATICS CLASSROOM[1)]

Heinrich Bauersfeld

> "What is a rite?" asked the little prince.
> "Those also are actions too often neglected", said the fox. "They are what make one day different from other days, one hour from other hours."
>
> Antoine de Saint Exupéry, 1943, p. 84

Abstract: Teaching and learning of mathematics in classrooms is interpreted as a situation of human interactions in institutionalized setting. Using theories and categories from different disciplines--e.g., ethnomethodology, linguistics, cross-cultural studies--a documented mathematics classroom episode is reanalyzed. The analysis of the example is used to identify four hidden dimensions in the classroom process and thus deficient areas of research: The constitution of meaning through human interaction, the impact of institutional settings, the development of personality, and the process of reducing classroom complexity. Consequences for teacher training towards more reflected classroom experience are drawn.

1. A nearly true story

In old Russia two men meet in a train somewhere between Moscow and Warsaw. Since the beaver collars indicate they are both merchants one of them asks: "Where are you going?" "To Moscow," the other replies. "Hey," says the first one, "if you say you go to Moscow you must really want me to believe that you go to Warsaw. But this train is headed for Moscow and this makes certain that you travel to Moscow. So, why are you lying to me?"

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1) This paper was developed for an invited address at the pre-session of the Special Interest Group for Research in Mathematics Education (SIG/RME) for the annual meeting of the National Council of Teachers in Mathematics (NCTM), Boston, April 18, 1979.

These two men are talking not only about directions, but more, they are concerned with their mutual expectations and with their subjective interpretation of what they "really" do. Though the replying man tells the "truth" from our contextual view, the asking protagonist understands the utterances as a lie.

Let us try an explanation. (This is not to kill a joke by explaining, but rather to use the explanation to illustrate a more important problem.) Competing merchants will hardly disclose good sources and addresses to each other. The questioner therefore expects a non-destination as answer. Knowing these rules of the interaction and hearing the obviously true destination he must construe a lie. Thus we laugh about a man, who seems to be the captive of his expectations. He became accustomed to this game and for him it is reality, his reality, Seen from a more general point of view our "truth" about the case is no better or more valid than is his "truth"--although we enjoy a larger majority supporting our interpretation.

This story about "situations", "rules", "expectations and interpretations", and "subjective realities" brings me directly to my theme: hidden dimensions in the so-called reality of a mathematics classroom. After a short overview of mathematics learning as a social activity and the role of related theory, the constitutive power of human interaction will be concretely demonstrated with a documented classroom situation. Following this, four deficient areas of research in mathematics education will be identified and discussed with a view to changing paradigms of research. Finally, I will come back to my main concern of pre-service and in-service teacher training and make some preliminary conclusions for it.

2. The Contribution of Social Sciences

To view the learning and teaching of mathematics as a social process--a "jointly produced social settlement" as Lee S. Shulman puts it (1979, also Bauersfeld, 1976)--seems to be a fairly recent issue. Although the ancient Greeks provided us with famous examples of mathematics instruction through dialogue (e.g., Plato's "Menon") we still do not have much information about the social dimensions of generating mathematical knowledge and of developing individual mathematical power within the classroom. Particularly researchers in mathematics education have not spent much time on these dimensions of human interaction. Other disciplines long since have produced relevant research specific to their disciplines although not to the learning of mathematics. Speaking of hidden or neglected dimensions within mathematics education is only relatively true in the sense that researchers have not made use of relevant developments in the other disciplines.

Examples of such contributions come from symbolic interactionism (Blumer, 1969; Goffman, 1969), linguistics (Gumperz, 1972; Herrlitz, 1977), ethnomethodology (Cicourel, 1974; Mehan, 1975 and 1979). The demarcation among these disciplines is difficult because of their increasing integration through interdisciplinary procedures--procedures which might also benefit mathematics education. Topics such as the generation of meaning and the function of language in social situations, the actual shaping of behavior and cognitive performance through human interaction, the specifity of communication in institutionalized settings, etc. apparently force interdisciplinary approaches and have formed a new type of human sciences.

There is a final point to make in this initial overview. From the very beginning my concern is both pragmatic, as well as highly theoretical. It is pragmatic since my goal is to improve mathematics teaching and learning through both teacher's and student's actions. It is theoretical because the "improvement"

and the "differentiated orientation" require the most sophisticated, reproducible theoretical framework available. Both aspects, the pragmatic and the theoretical, action and reflection, are deeply interwoven. Albert Einstein has put it sharply: "It is always the theory which decides what can be observed." (Mehra, 1973, p. 269). Though from a physicist one might expect to hear the complementary statement: It is always the observation (or the 'reality') which decides the theory. In the human sciences different actions and different concerns often produce different theories, or different theories in turn produce different realities.

This point is often expressed in education by saying that research findings, like a theory on certain classroom events, need special transformation into teaching practice; or, "that there is little direct connection between research and educational practice" (Fred Kerlinger, 1977, p. 5); or, "... what is good theory for one purpose is not a good theory for another" (Ernest Hilgard, 1976). - All these statements are only different expressions in an educational setting of the general point made by Einstein.

3. The Constitutive Power of Human Interaction

Two dissertations mark cornerstones for the discussion of human interaction in the mathematics classroom: George Bernard Shirk's "Examination of conceptual framework of beginning mathematics teachers" (1972), and Stanley Erlwanger's "Case studies of children's conception of mathematics" (1974). Both were directed by Jack A. Easley, University of Illinois, Urbana/Champaign.

Erlwanger's case studies are related to programs from Individual Prescribed Instruction. His documentation of students' mathematical misconceptions and deficiencies demonstrate how mathematics learning can be damaged by restricted teacher-student

communication--a restriction which leads to the nearly total absence of negotiations over meanings. It should be clear that the fading fascination of programmed instruction does not offer a satisfactory explanation for the non-appearence of further research with such case studies.

3.1 The episode. Shirk's work with beginning teachers gives a striking example of the influence of subjective theories about mathematics teaching, the student's role and teacher's role. Moreover, his documents give an idea of the fragility of classroom discourse and of the impact of these social situations on mathematics learning. Therefore, I will take a brief example from Shirk's transcripts and use it for comments based on theories from other human sciences.

The episode presents an early part of a beginning teacher's lesson with eighth graders at Urbana Junior High School. The topic is about slides, flips, and turns from "Motion Geometry" written by Russel Zwoyer and Jo McKeeby Phillips (a product of Max Beberman's UICSM). In the preceding lesson Tom, the teacher, has defined parallel lines in terms of slides. The lesson under discussion opens with student's working on positive examples. The episode which I am going to reanalyze starts with line 39 of the transcript. The teacher presents a counter-example, two intersecting lines (see Figure 1).

Figure 1

"Tom, tape of 4/4/72" (from G.B. Shirk, 1972, p. 173/174)

T - teacher, Tom K - student, Kevin R - student, Reggie

T: ... ... look at the next figure, right below it. Ya, ... now, are those two lines parallel?

K: Nope.

T: Why not?

K: They cross each other.

T: OK, but, uh, ... according to what I've said about parallel lines, what can't they do?

K: Can't cross them ... (?)

T: What?

K: Can't .. they won't, they won't .. I'd rather not .. (ID)

T: What Reggie?

R: Um.

K: ... They won't come together ..

T: OK, but what did I say, what did I say on the first figure? What could you do to get from one line to the other?

K: Slide.

T: OK. There's a slide arrow that'll go, .. that'll take one line into the other.

K: ... Arrow.

T: Right? Is there a slide arrow, ... on the second figure? .. Reggie? On the second figure, can you draw a slide arrow that'll go from one of these lines to the other?

R: (?) Not any more.

T: Like the ... a slide arrow, ... will that take the, ... will that go from one to the other?

R: I don't know.

T: Well, you remember what a slide arrow did?

R: Hm?

T: It, ... it moved a figure along that slide. Can you draw one that'll do that?

R: Oh.

T: OK, is that a slide arrow?

R: (?)

T: What is that? What, ... what did you draw?

73 R: A circle.
74 T: Well, you remember.
75 R: (laughter)
76 T: ... What that was called?
77 R: (?)
78 T: Does anybody remember ..
79 R: A rotating ...
80 T: OK, we called it a what? ..... A turn?
81 R: A turn arrow.
82 T: OK, so what you were starting to draw in was a turn arrow,
83 right? But I'm just talking about slides. Can you draw
84 a slide arrow? Just a straight line, a straight arrow,
85 that'll go from one, that'll take one line to the other?
86 (pause for students to work) ...

First I shall follow Shirk's own interpretation and then add my comments later:

> "In this episode, Tom was working for a compound goal which he wanted the students to reach: ... that they recognize the lines illustrated were not parallel and realize there was a reason for it through the definition of slides. The students did accomplish the first part of this goal; but when they invoked a reason other than that Tom sought, it was rejected by him as being inappropriate. The students weren't connecting the lessons together. Tom could only see this as a completely unexpected deficiency in the student's understanding of a lesson which he believed had been learned earlier, so he turned the lesson toward this deficiency in an effort to correct for it." (Shirk, 1972, p. 42).

> "The critical aspect centers around Tom's expectation that the students would know all of the consequences of the definition of 'parallel'. Not seeing this, Tom interpreted their problem as having to do with slides; and this bothered him for he believed that slides had been adequately covered and, therefore, the students should know them. He was also assuming that the students would appreciate everything that he said and therefore, the problem would have to lie elsewhere, i.e., in their more basic preparations which he thought had been covered earlier." (Ibid, p. 46).

For the re-analysis it is useful to note the major shifts in the student-teacher interpretations of the situation. The episode then splits into four parts.

Part I, lines 39 - 51: The teacher does not succeed in using the counter-example to infer that intersecting lines cannot be parallel (there is no slide arrow which would move the lines together). Unexpectedly he receives a much simpler answer, not invoking motion geometry concepts, "they cross each other" (line 43). Albeit correct, the teacher rejects the answer as inadequate "on the basis only that they should remember what he said in the previous example." (Ibid, p. 47).

The students become confused and uncertain as evidenced by K's stammering and brief withdrawal (line 48).

Part II, lines 52 - 64: "Tom is now directing the student's attention toward the drawing again in an effort to get them to see the connection between it and the slide." (Ibid, p. 48). Repeatedly he uses the key word "slide arrow".

The students try to guess the teacher's intentions. Their answers are short and cautious: "Slide" (line 54) and ".. arrow" (line 57). Their uncertainty increases. Thus under the teacher's pressing Reggie modifies his answers from "Not any more" (line 61) to "I don't know" (line 64). "With their initial efforts rejected, and Tom emphasizing slides, the students begin to look around for ways to slide the two lines together for there was nothing in the earlier portions of the lesson about 'no slide' or 'not parallel'." (Ibid, p. 48).

Part III, lines 65 - 86: Still the teacher has not given up his initial aim. His impulse "Well, you remember" (line 74) is an attempt "to get Reggie to put together the formal principles by pointing out to him that what he had drawn were turns rather than slides" (Ibid, p. 49). Reggie's "failure" (as seen from the teacher's eyes) justifies the causal ascription that the students have forgotten all about slides. Therefore the teacher begins to "reteach" the concept towards the end of this section.

The students, however, "in an effort to come up with the answer they thought Tom was looking for, (namely, a slide arrow) invented slide lines between the two intersecting lines." (Ibid, p. 49), so Reggie at line 69 (see Figure 2). Reggie's misinterpreting the teacher's question (lines 74 - 76) "What that was called?" is completely in line with his looking for slides. The drawings of the students in this part (see Figure 3) expose

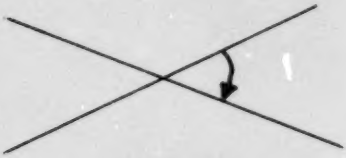

Figure 2
"Reggie at line 69"

Figure 3
"Students at line 85

the extent to which the teacher's pressing has contributed to "spoiling" the student's concept of "slide" and of "slide arrow". Nevertheless the teacher's interpretation that the students have not learned their slide lesson is not diminished but rather reinforced.

Part IV, lines 87 - 155: (This concluding portion of the episode is not included in the above quote.) "The remainder of this first episode involves Tom's attempts to reteach the concept of slides to the students." (Ibid, p. 51). And he works on that rehearsal until he gets the conviction that "he has reconnected the students with slides and parallel lines." (Ibid, p. 52).

On a more general level Shirk explains the episode using the terms "split personality" and "guessing ahead" (the latter from John Holt, 1964):

> "A 'split personality' ... occurs when the teacher is teaching one lesson and the students, in an effort in 'psych out' the teacher, are actually learning another ... The 'split personality' in the lesson occurred as a result of the conceptual frameworks which governed Tom's actions, rather than being a part of those frameworks themselves. Tom, acting in accordance with his frameworks, interpreted the student's

> behavior in a certain way and then acted in a manner consistent with his conceptual frameworks. For their part, the students attempted to guess ahead and therefore acted differently. These actions resulted in one path being taken by the students while Tom was trying to lead them along another." (Shirk, 1972, p. 43).

In the above analysis Shirk uses the classical relation of cause and effect as he matches causes with personal attributes. That is, he traces the outcomes back to properties and actions of single individuals. As a result he is lead to somewhat discouraging conclusions and recommendations. From his finding that "there was no change discernible within the conceptual frameworks" he concludes that the future teacher education programs "must be so designed so as to be assimilable to the preexisting conceptual frameworks" (Shirk, 1972, p. 165). I shall try an alternate answer to the teacher training problem later, but first let me give an interpretation of the episode from a different paradigm of social action.

A description of the situation as constituted through the interaction of the participants can challenge the usual causal model, cast doubt on predictive conclusions and possibly shed light on the use of language in the mathematics classroom.

3.2 The constitution of the social situation. Principally, and taken as a piece of an ongoing process, the episode cannot be reconstructed sufficiently--neither from personal variables, from characteristics of the single participant, or from the documented speech production. Hence, from additional interviews with the teacher, and from essays and "comment cards" which the teacher had to write, Shirk has distilled the teacher's conceptualization of mathematics education, of his role as a teacher, and of the student's role. Shirk uses this set of statements only to explain the teacher's moves.

We do not have any comparable information about the students. However, the interactive nature of the process and the mutual relatedness of expectations and interpretations can be partially reconstructed from the transcript. For the purpose of this analysis consider the following table.

Table 2

Interpretations and Changes

Aspects of analysis	Parts of the episode			
	I: lines 39 - 51	II: lines 52 - 64	III: lines 65 - 86	IV: lines 87 - 155
Teacher's interpretation	Use counter-example to strengthen concept. Diappointment about student's failure.	Prompting will help the students catch on. Increasing disappointment.	They don't really know what a slide is. Confusion.	Give up and reteach the concept of "slide". Resignation.
Student's interpretation	Having not treated counter-examples they think about easier descriptions.	There must be something to say or to do with "slide arrows".	He insists on arrows which connect the lines.	He wants us to play the "recitation game".
The actual task, teacher's view	Connect the non-existence of a slide to the non-parallelism of the lines.	Draw the student's attention to the role of the slide arrow.	Find out that there is no slide arrow for intersecting lines.	Help the students reconstruct and re-correct the concept of "slide".
The actual task, student's view	Tell if intersecting lines are parallel.	Find another description. Later confusion, "Don't know".	Look for new arrows which might match the intersecting lines.	Tell what you have learned previously-- "like we said it yesterday...".
Meaning of "slide arrow" for student	Move in positive examples one line onto the other	(Opening and changing.)	Any move, straight, or turn or up-and-down.	(Like in part I? but scarred and more divergent.)

The first two lines in Table 1 reconstruct the teacher's interpretation and the student's interpretation of the four parts. (A more detailed analysis can be developed through following the discussion step-by-step.) Comparing the mutual interpretations in columns gives a rough but sufficiently clear idea.

The teacher's immediate objectives change following his changing interpretations of the process. "Guessing ahead" the student's interpretation of the teacher's intentions changes as well. By no means are the actions of the two sides, teacher and students, reactions only to the preceding move of the other side. It is commonly believed that individuals react to the actions of another when in fact they react to their self-constructed interpretation. Yet, "reaction" is misleading. Far from the simple model of stimulus-response the participant's actions in this social situation are generated through complicated, internal reflective activity. This subjective reflective activity takes into account not only the actual and perceptable moves of the others but also the more general interpretations of the situation, and one's own role in that situation. Furthermore, actual interpretations of related former experiences exercise an influence on the current ongoing interpretation. Each participant's actions contribute to the change of the other's, of their interpretation and their actions. And through this process they contribute to the change of the participant's own interpretation and action. Thus it becomes reasonable to speak of the "constitution" of the social situation (Mehan, 1975). More precisely: The social situation is constituted at every moment through the interaction of reflective subjects. Ethnomethodologists therefore describe "reality as a reflexive activity" (Mehan & Wood, 1975, p. 8).

The episode under discussion is an example of the constitutive power of human interaction, i.e. the interaction constructs the subjects' various realities. Both teacher and students act according to their actual subjective realities. The students

draw turn arrows for slides; the teacher diagnoses learning deficiencies; and teacher and students work clearly at cross purposes convinced they understand the situation, clearly.

> "Every moment is mysterious, as the understood horizon of the moment is inexhaustible. Every interpretive act indexes this mystery in an unpredictable way. A person's every action is thus creative; it reflexively alters the world. The person begins with certain materials that set limits, and then acts and in acting alters those limits." (Mehan & Wood, 1975, p. 203).
>
> "Forms of life are always froms of life forming. Realities are always realities becoming." (Melvin Pollner, in Mehan, 1975, p. 32).

3.3 The constitution of meaning. Not only subjective interpretation and assessment change during the process but also the aims, the actual tasks, and even the concepts. A comparison of the teacher's and the student's view during the episode of the task (see Table 1 third and fourth line) emphasizes the semantic change of the problem situation. Clearly, each participant's view of the actual task to be done is different and varying during the course of the episode. The task must be understood as a function of the situation.

For the students the concept of "slide arrow" also varies across the episode (see Table 1 fifth line). In the beginning the previous experiences with parallel lines (and slide arrows moving them together) is dominant. The intervention of the counter-example and the following discussion with the teacher produce doubts about where to locate the borderlines of the concept. What is and what is not to be included? The student's interpretation of the teacher's insistent questions increasingly spoils the concept and leads to an arbitrary guesses as to the true meaning. Any type of arrow, curved or up-and-down, is used by the students (see Figure 2 and 3).

Without further information on the student's thinking the effect of the reteaching is difficult to evaluate. Surely the residual status of the concept "slide arrow" will differ from its initial status but it may not be improved. Due to the high affective load during part II, the concept now might be vulnerable to future misunderstanding in similar situations.

Thus, the logical principle of identity is not applicable. The word "slide arrow" does not mean the same to every participant. Moreover, the meaning changes during the episode repeatedly and remarkably. But, if problems and concepts become functions of the situation instead of being constant and stable, it then becomes necessary to consider the social constitution of meaning, i.e., the constitution of meaning through human interaction.

Herbert Blumer makes the same point: "Symbolic interactionism sees meanings as social products, as creations that are formed in and through the defining activities of people as they interact." (1969, p. 5). However, as a matter of principle, there is small chance of predicting the outcomes of such episodes at their beginning. Nor is there much chance of making predictions about a later stage from the basis of preceding one. Since we cannot ascribe the constitution of meaning to one single participant (e.g., the teacher), we are not in a position to use causal models as adequate descriptors of individual social interaction--particularly not of mathematics teaching and learning. "As every day meanings do not meet the canons of logic, they are transformed by literal description. These transformed meanings are amenable to causal models. Every day life is not." (Mehan, 1975, p. 66).

At this point the analysis leads into a revolution of fundamental paradigm (following Thomas S. Kuhn, 1962). In the human sciences rules are different than in the natural sciences. For example, they are rules about the constituting of situations

and meanings rather than rules about the situations and meanings themselves. They are rules about structuring the process rather than about structure of the process. (See Hugh Mehan's title "Structuring the school structure", 1978). In the human sciences the interpretation-assessment paradigm will replace the cause-effect paradigm borrowed from the natural sciences.

3.4 The role of language. The above episode also prompts a new look at the role of language in mathematics teaching. Years ago linguistic research would have used the utterances in Shirk's episode for a syntactical and a semantic analysis of the material. For example, the analysis might have pointed to a lack of adjectives. Or, it might have noted that the units of speech consist only of short or broken sentences, that paratactical structures dominate, or that many deictic words appear (words which demonstrate or point to something--"here", "this", "there", etc.). Without any additional information the analyst would speak of a "restricted code" and perhaps a resulting poorly developed meaning.

It is the fundamental idea in Chomsky's theory (1957) that we cannot reconstruct the constitutional process of communication from the surface, that is, from recorded speech. Neither "discovery-models" (for the discovery of new grammars) nor "decision-models" (for the decision about the adequacy of a grammar) will work. Chomsky maintains that these models only make use of the linguistic data. Since we have to analyze the rules of structuring communication we must analyze evaluative structures of individual interpretation and assessment and this analysis requires an "evaluation-model" (for the evaluation among existing grammars). At this point there is a change of paradigm from objective (linguistic) data to interpretative structures as the object of analysis. Since Chomsky's work sociolinguists have increasingly studied the use of language in social interaction.

An interesting and helpful issue is the concept of indexicality. If there is no further information, then most people will not understand the discourse in lines 41 - 51 of the episode:

41 K: Nope.
42 T: Why not?
43 K: They cross each other.
44 T: OK, but, uh, ... according to what I've said about
45 parallel lines, what can't they do?
46 K: Can't cross them ... (?)
47 T: What?
48 K: Can't .. they won't, they won't .. I'd rather not ... (ID)
49 T: What Reggie?
50 R: Um.
51 K: ... They won't come together ...

(Shirk, 1972, p. 173)

Mathematics educators may even become doubtful about the topic and meaning of the discussion if they don't know that the document is related to mathematics instruction. Bar-Hillel has defined such "utterances that require contextual information to be understood" as "indexical expressions". (quoted by Mehan, 1975, p. 93). Thus an "informed" outsider can even have difficulty understanding what is going on in a discussion. It is difficult to realize what the participants intend to say and to identify the meaning they create in the given situation. As a prerequisite of communication participants have to share common understandings which they take as an implicit basis of reference when speaking to each other. While speaking, each participant anticipates the understanding and the interests of the specific addressee. The speech gets organized through the expectation of what the addressed person already knows. Each speaker uses his interpretation of the given situation and of the addressee as an index from which he forms his utterances and from which he decides his "choice of grammar".

In the classroom the "teacher's instructions are indexical expressions which requires teachers and children to employ contextually bound interpretative practices to make sense of these instructions." (Cicourel, 1974, p. 129). What a participant says not only transports the intended message, but over and above the message the utterance contains information about his understanding of the topic, his interpretation of the situation, his expectations of what the others might know, as well as his present emotional concerns.

Hence, indexicality is another label for the thesis that the situation influences how language is used. Not only content and meanings are negotiated and constituted in the social situation, but also the use of language and the performance of the speaker are co-determined. This is true not only for the syntactical structure of the utterances but also for the choice of words. (From this point of view Bernstein's distinction (1965) between "elaborated" and "restricted" codes might be more an issue of the indexical and reflexive constitution of the situation rather than of the competence of the speakers.)

Mathematicians, in particular, have invested much effort in producing universal statements, and most school mathematicians would claim any mathematical statement as non-indexical, i.e., as universal and objective. However, this conviction blocks insight into the irreparable incompleteness of utterances, and more general, of any symbolic action. Each utterance, just as each symbolic form, is necessarily incomplete, because it has to be filled in with meaning via contextual interpretation. Through its genesis and chain of definition a concept inevitably gets infiltrated with contextual information. And "every attempt at repair increases the number of symbols that need to be repaired." (Mehan & Wood, 1975, p. 93). Therefore, understanding mathematics is not only a case of logic, or of divergent thinking, or of proper definitions. As far as understand-

ing is realized in social interaction (or through communication, which is the very same thing) it inescapably becomes dependent upon the interpretative, indexical, and reflexive constitution of meaning.

4. Four deficient areas of research--mathematics education's hidden dimensions

The process of teaching and learning mathematics can be viewed most aptly as a highly complex human interaction in an institutionalized setting--an interaction which forms a distinctive part of the participant's life. Four aspects in this issue deserve more detailed discussion since they represent weak areas of research.

4.1 Teaching and learning mathematics is realized through human interaction. It is a kind of mutual influencing, an interdependence of the actions of both teacher and student on many levels. It is not a unilateral sender-receiver relation. Inevitably the student's initial meeting with mathematics is mediated through parents, playmates, teachers. The student's reconstruction of mathematical meaning is a construction via social negotiation about what is meant and about which performance of meaning gets the teacher's (or the peer's) satisfaction "Symbolic interactionism sees meaning as social products, as creations that are formed in and through the defining activities of people as they interact." (Herbert Blumer, 1969, p. 5). How can we expect to find adequate information about teaching and learning when we neglect the interactive constitution of individual meanings?

4.2 Teaching and learning mathematics is realized in institutions which the society has set up explicitly to produce shared meanings among their members. Institutions are represented and

reproduced through their members and that is why they have characteristic impacts on human interactions inside of the institutional. They constitute norms and roles; they develop rituals in actions and in meanings; they tend to seclusion and self-sufficiency; and they even produce their own content--in this case, school mathematics. How reliable are studies on the effects of mathematics education if they do not take into account the institutional impact on teacher and student? The question becomes crucial when one thinks about any application of knowledge learned at school to situations outside of school.

4.3 Mathematics education constitutes a distinctive part of the student's life as well as of the teacher's. Anyone who is active in mathematics, will learn something about himself, especially since the activity happens in interactive situations. On the other hand one can learn mathematics only by actively engaging his previous knowledge of related subjects and actions. Therefore mathematics education is deeply related to the man-made world of symbols and meanings, to common sense, and to everyday life. Mathematics education depends on our social and historic conditions. How can we dare to make any prediction about the mathematical abilities of a student and about his chances to develop these abilities if not by carefully relating such statements to his personality and background?

4.4 Scientists are not the only ones who have difficulties dealing with highly complex issues. The orientation for actions and decisions in the classroom continuously requires the reduction of complexity. On the other hand the understanding and the effective reduction of complexities demands their total unfolding and complete reconstruction. To date scientific analysis has been incapable of reducing the complexities of an actual mathematics classroom sufficiently for guiding a teach-

er's decisions. Yet without such guidance in the process of reducing it is impossible to plan effective teacher training program.

5. Fundamental Problems of Research and Development

Within the last years my view of the structure of the classroom process has been changed as a result of collecting information from several disciplines, participating in mathematics lessons, and analyzing video-taped mathematics lessons. My subjective change includes the aims of my work, the subject to be studied, the methods of research, as well as the underlying paradigms of my thinking. This personal event is worth of mention since discussions with colleagues leads me to believe that my subjective difficulties only mirror much more fundamental difficulties within our profession. Philosophers of science agree that such difficulties within a profession are strong indicators of fundamental change of paradigms.

In the present transition stage three theses seem to be of importance:

5.1 Mathematics education is deeply in need of theoretical orientation. We have too much research on too small a theoretical basis. Many opening addresses of APA, AERA, and SIG/RME, as well as journal articles from within the last years have complained about this problem, e.g., Lee Cronbach (1975), Lee S. Shulman (1979), Ernest Hilgard (1976). Perhaps there are too many short-termed research contracts. Perhaps there is too much prescription for "acceptable" research programs. Or, perhaps, there is no support for methodological heretics and thus no encouragement for young researchers to try unusual approaches. For sure, there has yet to appear an adequate forum for theoretical discussion (compared with West German publications the United States provide for very little discussion of metatheo-

ry.) Whatever the cause, I do not believe that there are not enough new ideas.

5.2 Research and practical developments follow different paradigms. The main stream of research still follows the paradigm of the natural sciences, stating an objective educational reality, using well defined and quantifiable concepts, and analyzing the relationship among them through statistical means. For a long time we have heard and accepted complaints about the complete lack of classroom applicability of research results. (See Kerlinger, 1977, for a sum-up from a researcher's view.)

On the contrary, the majority of a teacher's classroom decisions are made via common sense and intuition rather than through rational analyses by scientific means. If she/he is a good teacher then her/his actions are based on a more differentiated perception of the classroom events than research recognizes. She/he is more open to contextual changes, "knowing" a student, using "tacit Knowledge" (M. Polanyi), and informal reflected experiences. Compared with these "hard" social facts, current research appears as "soft-ware". It is necessary that research in mathematics education takes notice of this gap if a claim for practical relevance is to be established.

5.3 Interdisciplinary approaches are promising, if not necessary, for closing mathematics education's "credibility gap." Within the broad area of social sciences the discrepancies between rationalistic and hermeneutic descriptions, between naive and scientific constructs have been realized and investigated much earlier. It is time to integrate these findings into our profession and to transform this knowledge to the specific conditions of learning and teaching mathematics.

Unlike the natural sciences the human sciences must deal with an objective social reality on the one hand, yet on the other hand must deal with as many realities as there are reflective subjects. Paradoxically, modern physicists have a highly developed understanding of explanatory models losing their meaning in the light of more comprehensive theories. For example, the question of the "divisibility" or the "consistency" of a light quantum (photon) makes no sense in a general theory of elementary particles because the theory describes the relations among elements but is not concerned with the nature of the elements themselves. This is very near to an important issue of constitutive ethnomethodology. The structuring activities of the participants form ("constitute") the social situation among themselves, and the process and rules of these "structurings" (as Mehan calls them) build a core theory of social action. The theory is related to structurings rather than to structures of the situation in usual sciences. That is, there is grater generalizability within the process of structurings than within the structures themselves.

6. Implications for Teacher Training

Those who find the discussed theories and interpretations more or less acceptable might find themselves forced to think about consequences. "It seems likely, that innovation in schools will not be of a very radical kind unless the categories teachers use to organize what they know about pupils and to determine what counts as knowledge undergo a fundamental change." (Keddie, 1971, p. 156). Shirk's findings about the stability of fundamental conceptualizations across teacher training apparently do not leave much chance for that change. How is it possible for a beginning teacher to overcome the sixteen or more years of his own experiences as a student? Particularly, since the contextual force of these experiences often dominates any later verbal information about education and leads

the beginner into an almost unconscious reproduction of the school system's characteristics.

But, if we form our cognition and behavior about teaching through social situations, then we can also change this formed cognition and behavior through social situations. We learn to behave in social settings only through the reflected participation and action in social settings. Similarly, a teacher will learn to teach or to change his teaching pattern only through reflected teaching. Yet, this is not the ruling model of present pre-service teacher training.

Usually the teacher student learns about teaching in contexts very different from classroom situations. The organizing interest for picking up knowledge in lectures is more along with passing examinations than related to later classroom application. Through various lectures and seminars the student teacher collects incomplete eclectic knowledge and she/he is left with the unassisted task of integrating this knowledge into an applicable system for the living classroom.

If the constitutive power fo social situations on behavior, meaning and language is as strong as assumed here, then the student teacher will have to spend much more time planning, accomplishing and reflecting upon real classroom teaching experience. From the very beginning the teacher to be must encounter an adequate complexity of social classroom exchanges. "Adequate" means that the complexity of the teaching-learning situation might be reduced in quantity, e.g., via a reduced number of students to teach or a reduced amount of lesson time, but not reduced in quality (as simulation games or video-tape analyses, e.g., would cause). If we claim to educate human beings, then a teacher will have to receive a much more careful, holistic preparation.

This, of course, will require support and development on the side of research as well. And this research, at least a reasonable part of it, will have to follow the interpretative paradigm. "Science at its best is thus like a firm but gentle hand that holds a butterfly without crushing it." (Kenneth S. Bowers, 1973, p. 332).

REFERENCE NOTES

1. Hilgard, Ernest R. Presentation at the Symposium on 75th Yearbook of NSSE, AERA annual meeting, April, 1976
2. Shulman, Lee S. Presentation at the SIG/RME pre-session of the NCTM annual meeting, Boston, April 18, 1979

REFERENCES

Bauersfeld, Heinrich (ed.): Fallstudien und Analysen zum Mathematikunterricht, Hannover: Schroedel Verlag, 1978

Bernstein, Basil. Class, codes, and control. London: Routledge and Kegan, Paul, 1973

Blumer, Herbert. Symbolic interactionism. Englewood Cliffs, New Jersey: Prentice-Hall, 1969

Bowers, Kenneth. Situationism in Psychology: An analysis and a critique. Psychological Review vol.80, September 1973, no. 5, p. 307-336

Cazden, Courtney. "You all gonna hafta listen". In: W.A. Collins (ed.): Children's language and communication. Hillsboro, New Jersey: Lawrence Erlbaum, 1979

Cicourel, Aaron et al. Language use and school performance. New York: Academic Press, 1974

Cicourel, Aaron. Cognitive sociology. Language and meaning in social interaction. Harmondsworth, England: Penguin, 1973

Cole, Michael; Gay, John; et al. The cultural context of learning and thinking. London: Methuen, 1971

Cole, Michael, and Scribner, Sylvia. Culture and thought. New York: Wiley, 1974

Erlwanger, Stanley Herbert. Case studies of children's conceptions of mathematics. Dissertation, 1974, University of Illinois, Urbana-Champaign (reprinted in: Journal of Children's Mathematical Behavior vol. 1, no. 3, summer 1975, p. 157-281)

Giorgi, Amadeo. Phenomenology and the foundations of psychology. In: Nebraska Symposium on Motivation 1975, University of Nebraska Press, Lincoln 1976, p. 281-348

Goeppert, Herma (ed.). Sprachverhalten im Unterricht. UTB 642, München, Wilhelm Fink, 1977

Goffman, Erving. Strategic Interaction. Philadelphia: University of Pennsylvania Press, 1969

Groeben, Norbert/Scheele, Brigitte. Argumente für eine Psychologie des reflexiven Subjekts. Darmstadt: Steinkopff, 1977

Gumperz, John I. & Hymes, J. (eds.) Direction in Sociolinguistics: The ethnography of communication. New York: Holt, Rinehart & Winston, 1972

Herrlitz, Wolfgang & Göttert, Karl-Heinz. Linguistische Propädeutik. Two vol's. Tübingen, West Germany: Max Niemeyer, 1977

Hoetker, James/Ahlbrand, William P., jr. The persistance of recitation. In: American Educational Research Journal, vol. 6, no. 2, March 1969, p. 145-167

Holt, John. How children fail. New York: Delta Book, 1964

Kerlinger, Fred. The influence of research on education practice. Educational Researcher, 1977, 6, no. 8, 5-12

Mehan, Hugh/Wood, Houston. The reality of ethnomethodology. New York: John Wiley & Sons, 1975

Mehan, Hugh. Structuring school structure. Harvard Educational Review, 1978, 48, p. 32-64

Mehan, Hugh. Learning lessons. Social organization in the classroom.Cambridge, Massachussetts: Harvard University Press,1979

Mehra, Jagdish. The physicist's conception of nature. Dordrecht: Reidel Publishing Company, 1973, (pp.264-275: Werner Heisenberg: Development of concepts in the history of quantum theory.)

Nisbett, Richard/DeCamp Wilson, Timothy. Telling more than we know: Verbal reports on mental processes. Psychological Review vol. 84, no. 3, May 1977, p. 231-259

Polanyi, Michael. The tacit dimension. Garden City, New York: Doubleday, 1966

Shirk, George Bernard. An examination of conceptual frameworks of beginning mathematical teachers. Dissertation, Urbana-Champaign: University of Illinois, 1972

Smith, Louis M. Anatomy of educational innovation: An organizational analysis of an elementary school. New York: John Wiley & Sons, 1971

Smith, Louis M./Geoffrey, William. The complexities of an urban classroom. An analysis toward a general theory of teaching. New York: Holt, Rinehart and Winston, 1968

UNESCO. New trends in mathematics teaching, vol. 4, Paris, 1979

Witz, Klaus. Group mind. Report no. 3, Committee on Culture and Cognition, Urbana-Champaign: University of Illinois, September 1978

Witz, Klaus. Fundamental change in school, society, and general consciousness. (part I). Report no. 2, Committee on Culture and Cognition, Urbana-Champaign, University of Illinois, August 1978

Walcott, Harry F. The man in the principal's office. An ethnography. New York: Holt, Rinehart and Winston, 1973

ADDITIONAL REFERENCES

Bernstein, Basil. A socio-linguistic approach to social learning. In: J. Gould (Ed.), Social Science Survey, 1965.

Cronbach, Lee. Beyond the two disciplines of scientific psychology. American Psychologist, 1975, vol. 30, p. 116 - 127.

Keddie, Neil. Classroom knowledge. In: M.F.D. Young (Ed.) Knowledge and Control. London: Collier-Macmillan, 1971.

Kuhn, Thomas S. The structure of scientific revolutions. Chicago: University of Chicago Press, 1962.

Smith, Louis M. An evolving logic of participant observation, educational ethnography and other case studies. In: Lee S. Shulman (Ed.) Review of Research in Education, vol. 6, Itasca: Peacock Publishers, 1978.